Процессы авторегрессии и скользящего среднего

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Autoregressive Process

Definition

Discrete-time autoregressive process of order p (AR(p) process) $\{Y_t, t = 0, 1, ...\}$ is defined as:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t, \quad t \geq p$$

where $\{\varepsilon_t, t = 0, 1, ...\}$ is a discrete-time white noise and $c, \phi_1, ..., \phi_p$ are constants

For AR(p) process $\{Y_t, t = 0, 1, ...\}$ the initial conditions $Y_0, ..., Y_{p-1}$ must be determined

The process $\{\varepsilon_t, t = 0, 1, ...\}$ is called innovation process and it usually is a Gaussian white noise with zero mean and variance σ^2

Y_t depends only on current and previous innovations $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...$ and for any $s > t$ the Y_t and ε_s are independent

AR(1) Process

AR(1) process:

$$Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t$$

Backward substitution:

$$\begin{aligned} Y_t &= c + \phi_1 Y_{t-1} + \varepsilon_t \\ &= c + \phi_1 (c + \phi_1 Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= c + \phi_1 (c + \phi_1 (c + \phi_1 Y_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1}) + \varepsilon_t = \ldots = \\ &= c\left(1 + \phi_1 + \phi_1^2 + \ldots + \phi_1^{t-1}\right) + \left(\varepsilon_t + \phi_1 \varepsilon_{t-1} + \ldots + \phi_1^{t-1} \varepsilon_1\right) + \phi_1^t Y_0 \\ &= c \sum_{i=0}^{t-1} \phi_1^i + \sum_{i=0}^{t-1} \phi_1^i \varepsilon_{t-i} + \phi_1^t Y_0 \end{aligned}$$

Y_t depends on all previous random variables $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ and initial state Y_0

Stability of AR(1) Process

AR(1) process: $Y_t = c\sum_{i=0}^{t-1}\phi_1^i + \sum_{i=0}^{t-1}\phi_1^i\varepsilon_{t-i} + \phi_1^t Y_0$

- $|\phi_1| < 1$:

$$Y_t = \frac{c}{1-\phi_1} + \sum_{i=0}^{t-1}\phi_1^i\varepsilon_{t-i} + \phi_1^t Y_0 \quad \text{(stable)}$$

- $\phi_1 = 1$:

$$Y_t = \sum_{i=1}^{t}(c+\varepsilon_i) + Y_0 \quad \text{(unstable, it's a random walk)}$$

- $|\phi_1| > 1$:

The sum $\sum_{i=0}^{t-1}\phi_1^i$ explodes exponentially (unstable)

Stationary Form of AR(1) Process

For $|\phi_1| < 1$: $\phi_1^t Y_0 \to 0$ as t grows, and the effect of initial condition Y_0 on Y_t will be small:

$$Y_t = \frac{c}{1-\phi_1} + \sum_{i=0}^{t-1} \phi_1^i \varepsilon_{t-i} + \phi_1^t Y_0 \approx \mu + \sum_{i=0}^{t-1} \phi_1^i \varepsilon_{t-i}$$

where $\mu = \dfrac{c}{1-\phi_1}$

"Infinite history" version of $AR(1)$-process:

$$Y_t = \mu + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}$$

It is stationary form of $AR(1)$-process (after the transient effect of initial condition Y_0)

Expectation of AR(1) Process with $|\phi_1| < 1$

AR(1) process in stationary form:

$$Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t = \mu + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}, \quad \mu = \frac{c}{1-\phi_1}$$

$$\mathrm{M}[Y_t] = \mathrm{M}\left[\mu + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}\right] = \mu + \sum_{i=0}^{\infty} \phi_1^i \mathrm{M}[\varepsilon_{t-i}] = \mu$$

Replacing c with $\mu(1-\phi_1)$ the process can be written w.r.t. deviations from the mean:

$$Y_t = \mu(1-\phi_1) + \phi_1 Y_{t-1} + \varepsilon_t$$

$$Y_t - \mu = \phi_1 (Y_{t-1} - \mu) + \varepsilon_t$$

$$\tilde{Y}_t = \phi_1 \tilde{Y}_{t-1} + \varepsilon_t$$

where $\tilde{Y}_t = Y_t - \mu$ is a centered random variable $Y_t, t = 0, 1, \ldots$

Autocovariance of AR(1) Process with $|\phi_1| < 1$

$$
\begin{aligned}
cov(t, t+\tau) &= \mathrm{M}[(Y_t - \mu)(Y_{t+\tau} - \mu)] = \mathrm{M}\left[\sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i} \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t+\tau-i}\right] \\
&= \mathrm{M}\left[\sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i} \left(\sum_{i=0}^{\tau-1} \phi_1^i \varepsilon_{t+\tau-i} + \sum_{i=\tau}^{\infty} \phi_1^i \varepsilon_{t+\tau-i}\right)\right] \\
&= \mathrm{M}\left[\sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i} \left(\sum_{i=0}^{\tau-1} \phi_1^i \varepsilon_{t+\tau-i} + \sum_{i=0}^{\infty} \phi_1^{\tau+i} \varepsilon_{t-i}\right)\right] \\
&= \mathrm{M}\left[\sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i} \sum_{i=0}^{\tau-1} \phi_1^i \varepsilon_{t+\tau-i}\right] + \phi_1^\tau \mathrm{M}\left[(Y_t - \mu)^2\right] \\
&= 0 + \phi_1^\tau \mathrm{D}[Y_t] = \phi_1^\tau \mathrm{D}\left[\sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}\right] = \phi_1^\tau \frac{\sigma^2}{1 - \phi_1^2}
\end{aligned}
$$

Autocovariance depends only on time shift τ

Autocovariance of AR(1) Process with $|\phi_1| < 1$. Notes

For AR(1) process with $|\phi_1| < 1$ the autocovariance function

$$c(\tau) = cov(t, t+\tau) = \phi_1^{\tau} \frac{\sigma^2}{1-\phi_1^2} = \phi_1 c(\tau - 1)$$

decreases to zero geometrically with factor ϕ_1:

$$cov(t, t+\tau) \to 0, \quad \tau \to \infty$$

but never equals to zero $c(\tau) \neq 0$ for all $\tau = 0, 1, \ldots$

- $0 < \phi_1 < 1$:
  y_t is similar to y_{t-1} due to the positive dependence, thus the graph of the time series evolves smoothly
- $-1 < \phi_1 < 0$:
  y_t in general is the opposite sign of y_{t-1}, thus the graph shows many changes of signs

Stationarity of AR(1) Process

- $|\phi_1| < 1$:

$$\mathrm{M}[Y_t] = \frac{c}{1-\phi_1} = \mu, \quad cov(t, t+\tau) = c(\tau) = \phi_1^\tau \frac{\sigma^2}{1-\phi_1^2}$$

$\Rightarrow \{Y_t\}$ is WSS process

- $|\phi_1| = 1$:

$$\mathrm{M}[Y_t] = ct \to \infty, \quad cov(t, t+\tau) = \sigma^2 t \to \infty, \quad t \to \infty$$

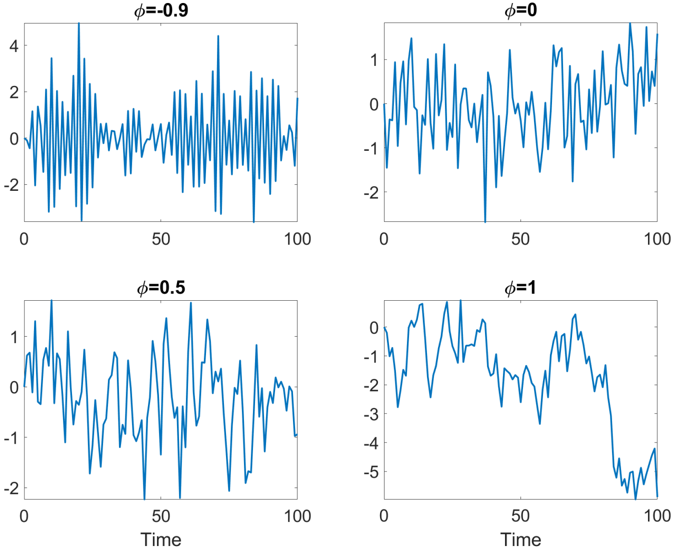
$\Rightarrow \{Y_t\}$ is non-stationary process (random walk)

- $|\phi_1| > 1$:

$$\mathrm{M}[Y_t] \to \infty, \quad \mathrm{D}[Y_t] \to \infty, \quad t \to \infty$$

$\Rightarrow \{Y_t\}$ is non-stationary exploding process

AR(1) Processes. Examples



Lag Operator

Definition

Lag (back-shift) operator L applied to random variable Y_t from discrete-time process $\{Y_t\}$ is defined as:

$$LY_t = Y_{t-1}$$

Lag operator can be applied multiple times:

$$L^2 Y_t = L(LY_t) = LY_{t-1} = Y_{t-2}$$

$$L^k Y_t = Y_{t-k}$$

The inverse operator L^{-1} is a forward-shift operator such that $L^{-1}L = 1$, where 1 is identity operator:

$$Y_t = 1Y_t = L^{-1}LY_t = L^{-1}Y_{t-1}$$

Autoregressive Operator

AR(p) process:

$$Y_t = c + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \varepsilon_t$$

In operator form (using the lag operator L):

$$Y_t - \phi_1 Y_{t-1} - ... - \phi_p Y_{t-p} = c + \varepsilon_t$$

$$(1 - \phi_1 L - ... - \phi_p L^p) Y_t = c + \varepsilon_t$$

$$\phi(L) Y_t = c + \varepsilon_t$$

where $\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$ is called as characteristic polynomial (or autoregressive operator) of AR(p) process

The autoregressive operator $\phi(L)$ can be viewed as a whitening operator. When applied to process $\{Y_t\}$ it gives white noise $\{\varepsilon_t\}$

The characteristic equation of AR(p) process:

$$1 - \phi_1 z - \phi_2 z^2 - ... - \phi_p z^p = 0$$

Inverse Autoregressive Operator of AR(1) Process

"Infinite history" version of AR(1) process can be derived from its operator form:

$$(1-\phi_1 L)Y_t = c + \varepsilon_t$$

$$Y_t = (1-\phi_1 L)^{-1}(c + \varepsilon_t)$$

Let's rewrite inverse operator $(1-\phi_1 L)^{-1}$ using series expansion:

$$(1-\phi_1 L)^{-1} = \frac{1}{1-\phi_1 L} = 1 + \phi_1 L + \phi_1^2 L^2 + \ldots \quad (|\phi_1| < 1)$$

$$\begin{aligned} Y_t &= (1 + \phi_1 L + \phi_1^2 L^2 + \ldots)(c + \varepsilon_t) \\ &= c + \varepsilon_t + \phi_1(c + \varepsilon_{t-1}) + \phi_1^2(c + \varepsilon_{t-2}) + \ldots \\ &= \frac{c}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i} = \mu + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i} \end{aligned}$$

Look-Forward Form of AR(1) Process

The series expansion of inverse autoregressive operator $(1-\phi_1 L)^{-1}$ can be applied only if $|\phi_1| < 1$

How to construct "infinite history" version of $AR(1)$-process $Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t$ for $|\phi_1| > 1$?

Let's rewrite $AR(1)$-process in look-forward form:

$$Y_{t-1} = -\frac{c}{\phi_1} + \frac{1}{\phi_1} Y_t - \frac{1}{\phi_1}\varepsilon_t$$

$$Y_{t-1} = \tilde{c} + \tilde{\phi}_1 Y_t + \tilde{\varepsilon}_t$$

where

$$\tilde{\phi}_1 = \frac{1}{\phi_1}, \quad |\tilde{\phi}_1| < 1$$

$$\tilde{c} = -\frac{c}{\phi_1}, \quad \tilde{\varepsilon}_t = -\frac{1}{\phi_1}\varepsilon_t$$

Stationary Form of $AR(1)$-process with $|\phi_1| > 1$

$AR(1)$-process in look-forward form:

$$Y_t = \tilde{c} + \tilde{\phi}_1 Y_{t+1} + \tilde{\varepsilon}_{t+1}$$

To solve this process we use a forward-shift operator $F = L^{-1}$:

$$(1 - \tilde{\phi}_1 F) Y_t = \tilde{c} + \tilde{\varepsilon}_{t+1}$$

As soon as $|\tilde{\phi}_1| < 1$:

$$\begin{aligned} Y_t &= (1 - \tilde{\phi}_1 F)^{-1}(\tilde{c} + \tilde{\varepsilon}_{t+1}) = (1 + \tilde{\phi}_1 F + \tilde{\phi}_1^2 F^2 + \ldots)(\tilde{c} + \tilde{\varepsilon}_{t+1}) \\ &= \tilde{c} + \tilde{\varepsilon}_{t+1} + \tilde{\phi}_1(\tilde{c} + \tilde{\varepsilon}_{t+2}) + \tilde{\phi}_1^2(\tilde{c} + \tilde{\varepsilon}_{t+3}) + \ldots \\ &= \frac{\tilde{c}}{1 - \tilde{\phi}_1} + \frac{1}{\tilde{\phi}_1} \sum_{i=1}^{\infty} \tilde{\phi}_1^i \tilde{\varepsilon}_{t+i} = \frac{c}{1 - \phi_1} - \sum_{i=1}^{\infty} \phi_1^{-i} \varepsilon_{t+i} \end{aligned}$$

Y_t depends on all future innovations $\varepsilon_{t+1}, \varepsilon_{t+2}, \ldots$

Causality of AR Process

Definition

The AR process $\{Y_t\}$ is called causal if it has a stationary representation (in terms of the white noise $\{\varepsilon_t\}$) such that Y_t depends only on ε_s, $s \leq t$, and doesn't depend on ε_s, $s > t$, for all $t = 0, 1, ...$

Non-causal processes are practically useless. For non-causal process it's necessary to know the future values $\varepsilon_{t+1}, \varepsilon_{t+2}, ...$ to calculate y_t

$AR(1)$-process with $|\phi_1| < 1$ is causal: $Y_t = \mu + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}$

$AR(1)$-process with $|\phi_1| > 1$ is non-causal: $Y_t = \mu - \sum_{i=1}^{\infty} \phi_1^{-i} \varepsilon_{t+i}$

$AR(1)$-process with $|\phi_1| = 1$ doesn't have stationary representation

Formal Definition of Causality

Definition

The AR(p) process $\phi(L)Y_t = c + \varepsilon_t$, where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p$$

is causal if its inverse autoregressive operator

$$\psi(L) = \phi^{-1}(L) = 1 + \psi_1 L + \psi_2 L^2 + \ldots$$

has absolutely summable coefficients $\sum\limits_{i=0}^{\infty} |\psi_i| < \infty$.

Absolute summability of $\{\psi_1, \psi_2, \ldots\}$ implies that Y_t will be finite:

$$Y_t = \psi(L)(c + \varepsilon_t) = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i} < \infty \quad (\psi_0 = 1)$$

Causality of AR(1) Process

Autoregressive operator of AR(1) process:

$$\phi(L) = 1 - \phi_1 L$$

Inverse autoregressive operator:

$$\psi(L) = 1 + \phi_1 L + \phi_1^2 L^2 + \ldots$$

Coefficients $\psi_0 = 1,\ \psi_1 = \phi_1,\ \psi_2 = \phi_1^2, \ldots$

- $|\phi_1| < 1$: $1 + |\phi_1| + |\phi_1|^2 + \ldots = \frac{1}{1-|\phi_1|} < \infty$
  $\Rightarrow$ AR(1) process is causal
- $|\phi_1| > 1$: $1 + |\phi_1| + |\phi_1|^2 + \ldots \to \infty$
  $\Rightarrow$ AR(1) process is non-causal
- $|\phi_1| = 1$: AR(1) process doesn't have stationary representation

Criterion of Stationarity and Causality

Theorem

The AR(p) process $Y_t = c + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \varepsilon_t$ is stationary, causal and ergodic iff

$$|z_i| > 1 \quad \forall i = 1, ..., p$$

where $z_1, ..., z_p$ are roots of its characteristic polynomial

$$\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p$$

AR(p) process is stationary $\Leftrightarrow$ all roots lie outside the unit circle

For AR(1) process:

Characteristic polynomial: $\phi(z) = 1 - \phi_1 z$, its root $z = \frac{1}{\phi_1}$

Stationarity condition: $\frac{1}{|\phi_1|} > 1 \Rightarrow |\phi_1| < 1$

Factorization of Autoregressive Operator

Characteristic polynomial of AR(p) process:

$$\phi(z) = -\phi_p \prod_{i=1}^{p} (z - z_i) = -\phi_p \prod_{i=1}^{p} z_i \left(\frac{z}{z_i} - 1 \right)$$

$$= -\phi_p (-1)^p z_1 ... z_p \prod_{i=1}^{p} \left(1 - \frac{z}{z_i} \right) = \prod_{i=1}^{p} \left(1 - \frac{z}{z_i} \right)$$

If all roots $|z_i| > 1$, $i = 1, ..., p$, then the inverse autoregressive operator can be represented as

$$\psi(L) = \phi^{-1}(L) = \frac{1}{1 - \phi_1 L - ... - \phi_p L^p} = \frac{1}{\prod\limits_{i=1}^{p} \left(1 - \frac{1}{z_i} L \right)}$$

$$= \prod_{i=1}^{p} \left(1 + \frac{1}{z_i} L + \frac{1}{z_i^2} L^2 + ... \right) = 1 + \psi_1 L + \psi_2 L^2 + ...$$

Stationary Form of AR(p) Process

The coefficients $\psi_1, \psi_2, \ldots$ can be obtained by imposing the cancellation of powers of L in identity equation $\psi(L)\phi(L) = 1$:

$$(1 + \psi_1 L + \psi_2 L^2 + \ldots)(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) = 1$$

$$\psi_1 = \phi_1$$

$$\psi_2 = \phi_1 \psi_1 + \phi_2$$

$$\psi_k = \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2} + \ldots + \phi_k = \sum_{i=1}^{k} \phi_i \psi_{k-i} \quad (\psi_0 = 1)$$

AR(p) process in operator form:

$$\phi(L) Y_t = c + \varepsilon_t$$

AR(p) process in stationary form:

$$Y_t = (1 + \psi_1 L + \psi_2 L^2 + \ldots)(c + \varepsilon_t) = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$$

Expectation and Autocovariance of Stationary AR(p) Process

For stationary AR(p) process $\{Y_t\}$ the expectation $\mathrm{M}[Y_t] = \mu = const$ for all $t = 0, 1, ...$:

$$\mathrm{M}[Y_t] = c + \phi_1 \mathrm{M}[Y_{t-1}] + ... + \phi_p \mathrm{M}[Y_{t-p}] + \mathrm{M}[\varepsilon_t]$$

$$\mu = c + \phi_1 \mu + ... + \phi_p \mu + 0$$

$$\mu = \frac{c}{1 - \phi_1 - ... - \phi_p}$$

It can be shown that autocovariance function $c(\tau)$ is

$$c(\tau) = \sum_{i=1}^{p} A_i \left(\frac{1}{z_i}\right)^{\tau}$$

where $A_1, ..., A_p$ are some constants and $z_1, ..., z_p$ are roots of characteristic polynomial $\phi(z)$

For stable process all roots $|z_i| > 1 \Rightarrow c(\tau) \to 0, \tau \to \infty$

Yule-Walker Equations

Autocovariances of stationary AR(p) process can be obtained using Yule-Walker method. It consists in multiplying centered process by $\tilde{Y}_t, \tilde{Y}_{t-1}, ...$ and taking expectations:

$$\tilde{Y}_t = \phi_1 \tilde{Y}_{t-1} + ... + \phi_p \tilde{Y}_{t-p} + \varepsilon_t$$

$$\mathrm{M}[\tilde{Y}_{t-\tau}\tilde{Y}_t] = \phi_1 \mathrm{M}[\tilde{Y}_{t-\tau}\tilde{Y}_{t-1}] + ... + \phi_p \mathrm{M}[\tilde{Y}_{t-\tau}\tilde{Y}_{t-p}] + \mathrm{M}[\tilde{Y}_{t-\tau}\varepsilon_t]$$

$$c(0) = \phi_1 c(1) + ... + \phi_p c(p) + \sigma^2$$

$$c(\tau) = \phi_1 c(\tau - 1) + ... + \phi_p c(\tau - p), \quad k > 0$$

The system of linear equations for $\tau = 1, ..., p$ is called as Yule-Walker equations:

$$\begin{aligned}
c(1) &= \phi_1 c(0) + \phi_2 c(1) + ... + \phi_p c(p-1) \\
c(2) &= \phi_1 c(1) + \phi_2 c(0) + ... + \phi_p c(p-2) \\
&... \\
c(p) &= \phi_1 c(p-1) + \phi_2 c(p-2) + ... + \phi_p c(0)
\end{aligned}$$

Yule-Walker Equations in Matrix Form

Defining:

$$c = (c(1), ..., c(p))^T \quad \text{is a vector of autocovariances}$$

$$\phi = (\phi_1, ..., \phi_p)^T \quad \text{is a vector of autoregression coefficients}$$

$$R = \begin{pmatrix} c(0) & c(1) & ... & c(p-1) \\ c(1) & c(0) & ... & c(p-2) \\ ... & ... & ... & ... \\ c(p-1) & c(p-2) & ... & c(0) \end{pmatrix}$$

is a matrix of autocovariances

Yule-Walker equations

$$R\phi = c$$

can be used to determine autoregression parameters:

$$\phi = R^{-1}c$$

Yule-Walker Equations for AR(1) and AR(2) Processes

Yule-Walker equations for AR(1) process:

$$\tilde{Y}_t = \phi_1 \tilde{Y}_{t-1} + \varepsilon_t$$

$$\begin{aligned} c(0) &= \phi_1 c(1) + \sigma^2 \\ c(1) &= \phi_1 c(0) \end{aligned} \quad \Rightarrow \quad c(0) = \frac{\sigma^2}{1-\phi_1^2}$$

$$c(\tau) = \phi_1 c(\tau - 1), \quad \tau > 0$$

Yule-Walker equations for AR(2) process:

$$\tilde{Y}_t = \phi_1 \tilde{Y}_{t-1} + \phi_2 \tilde{Y}_{t-2} + \varepsilon_t$$

$$\begin{aligned} c(0) &= \phi_1 c(1) + \phi_2 c(2) + \sigma^2 \\ c(1) &= \phi_1 c(0) + \phi_2 c(1) \\ c(2) &= \phi_1 c(1) + \phi_2 c(0) \end{aligned} \quad \Rightarrow c(0) = \frac{\sigma^2 (1-\phi_2)(1+\phi_2)^{-1}}{(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}$$

$$c(\tau) = \phi_1 c(\tau - 1) + \phi_2 c(\tau - 2), \quad \tau > 0$$

Ergodicity of AR(p) Process

Autocovariance function of stationary AR(p) process:

$$c(\tau) = \sum_{i=1}^{p} A_i \left(\frac{1}{z_i}\right)^{\tau}, \quad (|z_i| > 1,\ i = 1, ..., p)$$

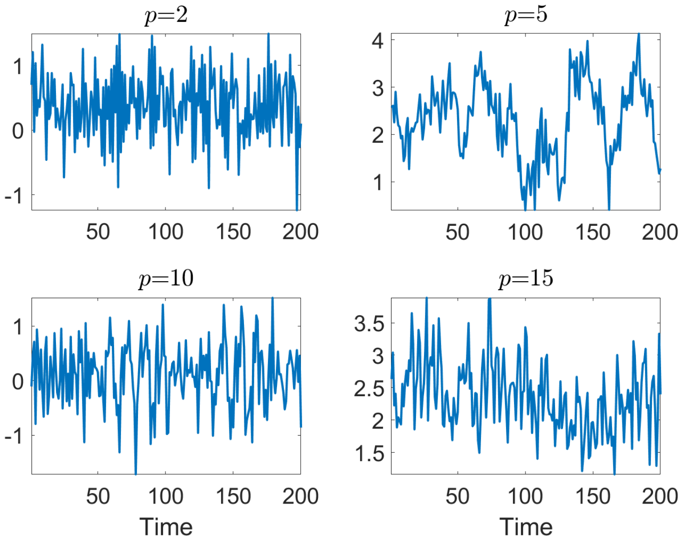
$$\frac{1}{T}\sum_{\tau=1}^{T} c(\tau) < \frac{A}{T}\sum_{\tau=1}^{T}\left(\frac{1}{z_{\min}}\right)^{\tau} \to \frac{A}{T}\frac{1}{1 - z_{\min}^{-1}} \to 0$$

Thus, stationary AR(p) process is mean-ergodic (by Slutsky's theorem)

It can be shown that stationary AR(p) process is also autocovariance-ergodic

If it is known that a real world process is stationary AR process then it is ergodic. Just one realization is needed to estimate its mean and covariance function

AR(p) Processes. Examples



Moving Average Process

Definition

Discrete-time moving average process of order q (MA(q) process) $\{Y_t, t = 0, 1, ...\}$ is defined as:

$$Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}, \quad t = 0, 1, ...$$

where $\{\varepsilon_t, t = -q, -q+1, ...\}$ is a discrete-time white noise and $c, \theta_1, ..., \theta_q$ are constants

The process $\{\varepsilon_t\}$ is innovation process and it usually is a Gaussian white noise with zero mean and variance σ^2

For MA(q) process $\{Y_t, t = 0, 1, ...\}$ the innovations at time moments $-q, -q+1, ..., -1$ must be determined

Y_t depends only on finite set of innovations $\varepsilon_t, \varepsilon_{t-1}, ..., \varepsilon_{t-q}$ and for any $s < t - q$ or $s > t$ the Y_t and ε_s are independent

Moving Average Operator

MA(q) process in operator form:

$$Y_t = c + (1 + \theta_1 L + ... + \theta_q L^q)\varepsilon_t = c + \theta(L)\varepsilon_t$$

where

$$\theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$$

is its characteristic polynomial (or moving average operator)

MA(q) process is always stationary, as it is a sum of stationary processes

Expectation of MA(q) process: $\mu = \mathrm{M}[Y_t] = c$ for all $t = 0, 1, ...$

Replacing c with μ the process $\{Y_t, t = 0, 1, ...\}$ can be written w.r.t. deviations from the mean:

$$\tilde{Y}_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$$

where $\tilde{Y}_t = Y_t - \mu$ is a centered random variable $Y_t, t = 0, 1, ...$

Autocovariance of MA(q) process

Centered MA(q) process: $\tilde{Y}_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$

Multiplying this expression by $\tilde{Y}_{t-\tau}$, $\tau \geq 0$, and taking expectations, the autocovariances are obtained:

$$\mathrm{M}[\tilde{Y}_t \tilde{Y}_{t-\tau}] = \mathrm{M}[\varepsilon_t \tilde{Y}_{t-\tau}] + \theta_1 \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-\tau}] + ... + \theta_q \mathrm{M}[\varepsilon_{t-q} \tilde{Y}_{t-\tau}]$$

$$\tilde{Y}_{t-\tau} = \varepsilon_{t-\tau} + \theta_1 \varepsilon_{t-\tau-1} + ... + \theta_q \varepsilon_{t-\tau-q}$$

$$c(\tau) = \begin{cases} (1 + \theta_1^2 + ... + \theta_q^2)\sigma^2, & \tau = 0 \\ (\theta_\tau + \theta_{\tau+1}\theta_1 + ... + \theta_q \theta_{q-\tau})\sigma^2, & \tau = 1, ..., q \\ 0, & \tau > q \end{cases}$$

MA(q) process has exactly the first q coefficients of the autocovariance function different from zero

MA(q) processes are always mean-ergodic (by Slutsky's theorem). It can be shown that they are also autocovariance-ergodic

Autocovariance of MA(1) process

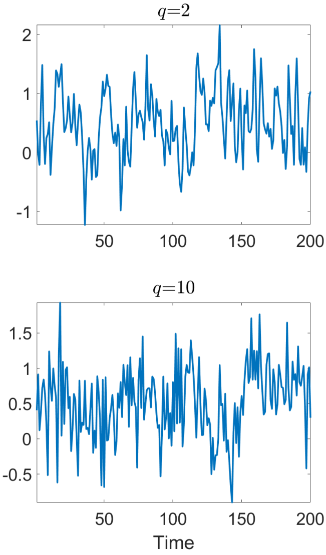
Centered MA(1) process: $\tilde{Y}_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$

$$
\begin{aligned}
c(0) &= \mathrm{M}[\tilde{Y}_t \tilde{Y}_t] = \mathrm{M}[\varepsilon_t \tilde{Y}_t] + \theta_1 \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_t] \\
&= \mathrm{M}[\varepsilon_t(\varepsilon_t + \theta_1 \varepsilon_{t-1})] + \theta_1 \mathrm{M}[\varepsilon_{t-1}(\varepsilon_t + \theta_1 \varepsilon_{t-1})] \\
&= \sigma^2 + \theta_1^2 \sigma^2
\end{aligned}
$$

$$
\begin{aligned}
c(1) &= \mathrm{M}[\tilde{Y}_t \tilde{Y}_{t-1}] = \mathrm{M}[\varepsilon_t \tilde{Y}_{t-1}] + \theta_1 \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-1}] \\
&= \mathrm{M}[\varepsilon_t(\varepsilon_{t-1} + \theta_1 \varepsilon_{t-2})] + \theta_1 \mathrm{M}[\varepsilon_{t-1}(\varepsilon_{t-1} + \theta_1 \varepsilon_{t-2})] \\
&= \theta_1 \sigma^2
\end{aligned}
$$

$$
\begin{aligned}
c(2) &= \mathrm{M}[\tilde{Y}_t \tilde{Y}_{t-2}] = \mathrm{M}[\varepsilon_t \tilde{Y}_{t-2}] + \theta_1 \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-2}] \\
&= \mathrm{M}[\varepsilon_t(\varepsilon_{t-2} + \theta_1 \varepsilon_{t-3})] + \theta_1 \mathrm{M}[\varepsilon_{t-1}(\varepsilon_{t-2} + \theta_1 \varepsilon_{t-3})] \\
&= 0
\end{aligned}
$$

MA(q) Processes. Examples



MA(1) Process

MA(1) process:

$$Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Y_t depends only on current ε_t and previous ε_{t-1} innovations

Let's rewrite it as a function of its previous values using backward substitution of innovations:

$$\varepsilon_t = -c - \theta_1 \varepsilon_{t-1} + Y_t$$

$$\begin{aligned} Y_t &= c + \varepsilon_t + \theta_1 \varepsilon_{t-1} = c + \varepsilon_t + \theta_1(-c - \theta_1 \varepsilon_{t-2} + Y_{t-1}) \\ &= c + \varepsilon_t + \theta_1(-c - \theta_1(-c - \theta_1 \varepsilon_{t-3} + Y_{t-2}) + Y_{t-1}) \\ &= c(1 - \theta_1 + \theta_1^2 - \dots) + (\theta_1 Y_{t-1} - \theta_1^2 Y_{t-2} + \dots) - (-\theta_1)^t \varepsilon_0 + \varepsilon_t \\ &= c \sum_{i=0}^{t-1} (-\theta_1)^i - \sum_{i=1}^{t-1} (-\theta_1)^i Y_{t-i} - (-\theta_1)^t \varepsilon_0 + \varepsilon_t \end{aligned}$$

AR(∞) Form of MA(1) Process

For $|\theta_1| < 1$: $(-\theta_1)^t \varepsilon_0 \to 0$ as t grows, and the effect of initial innovation ε_0 on Y_t will be small:

$$Y_t = \frac{c}{1+\theta_1} - \sum_{i=1}^{\infty} (-\theta_1)^i Y_{t-i} + \varepsilon_t$$

It is infinite-order autoregressive form of MA(1) process (after the transient effect of initial innovation)

Y_t depends on all previous variables $Y_{t-1}, Y_{t-2}, \ldots$

If $|\theta_1| < 1$ the effect of earlier variables Y_{t-i} on Y_t tends geometrically to zero with i

If $|\theta_1| > 1$ it produces the paradoxical situation in which the earlier Y_{t-i} the more effect it has on Y_t

Inverse Moving Average Operator of MA(1)-process

The AR(∞)-form of MA(1) process can be derived from its operator form:

$$Y_t = c + (1 + \theta_1 L)\varepsilon_t$$
$$(1 + \theta_1 L)^{-1}(Y_t - c) = \varepsilon_t$$

Let's rewrite operator $(1 + \theta_1 L)^{-1}$ using series expansion:

$$(1 + \theta_1 L)^{-1} = 1 - \theta_1 L + \theta_1^2 L^2 - ... \quad (|\theta_1| < 1)$$
$$(1 - \theta_1 L + \theta_1^2 L^2 - ...)(Y_t - c) = \varepsilon_t$$
$$(Y_t - c) + (-\theta_1 L + \theta_1^2 L^2 - ...)(Y_t - c) = \varepsilon_t$$
$$\begin{aligned} Y_t &= c - (-\theta_1 L + \theta_1^2 L^2 - ...)(Y_t - c) + \varepsilon_t \\ &= c + \theta_1(Y_{t-1} - c) - \theta_1^2(Y_{t-2} - c) + ... + \varepsilon_t \\ &= \frac{c}{1 + \theta_1} - \sum_{i=1}^{\infty}(-\theta_1)^i Y_{t-i} + \varepsilon_t \end{aligned}$$

Look-Forward Form of MA(1)-process

The series expansion of inverse moving average operator $(1+\theta_1 L)^{-1}$ can be applied only if $|\theta_1| < 1$

How to construct AR(∞) form of MA(1) process $Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$ for $|\theta_1| > 1$?

Let's rewrite MA(1) process in look-forward form:

$$\frac{1}{\theta_1} Y_t = \frac{c}{\theta_1} + \frac{1}{\theta_1}\varepsilon_t + \varepsilon_{t-1}$$

$$\varepsilon_{t-1} = -\frac{c}{\theta_1} - \frac{1}{\theta_1}\varepsilon_t + \frac{1}{\theta_1} Y_t$$

$$\varepsilon_{t-1} = -\tilde{c} - \tilde{\theta}_1 \varepsilon_t + \tilde{Y}_t$$

where

$$\tilde{\theta}_1 = \frac{1}{\theta_1}, \quad |\tilde{\theta}_1| < 1, \quad \tilde{c} = \frac{c}{\theta_1}, \quad \tilde{Y}_t = \frac{1}{\theta_1} Y_t$$

Stationary Form of MA(1) Process with $|\theta_1| > 1$

MA(1) process in look-forward form:

$$\varepsilon_{t-1} = -\tilde{c} - \tilde{\theta}_1 \varepsilon_t + \tilde{Y}_t$$

To solve this process we use a forward-shift operator $F = L^{-1}$:

$$(1 + \tilde{\theta}_1 F)\varepsilon_{t-1} = \tilde{Y}_t - \tilde{c}$$

$$\begin{aligned}
\varepsilon_{t-1} &= (1 + \tilde{\theta}_1 F)^{-1}(\tilde{Y}_t - \tilde{c}) = (1 - \tilde{\theta}_1 F + \tilde{\theta}_1^2 F^2 - \dots)(\tilde{Y}_t - \tilde{c}) \\
&= \tilde{Y}_t - \tilde{c} + (-\tilde{\theta}_1 F + \tilde{\theta}_1^2 F^2 - \dots)(\tilde{Y}_t - \tilde{c})
\end{aligned}$$

$$\begin{aligned}
\tilde{Y}_t &= \tilde{c} - (-\tilde{\theta}_1 F + \tilde{\theta}_1^2 F^2 - \dots)(\tilde{Y}_t - \tilde{c}) + \varepsilon_{t-1} \\
&= \tilde{c} + \tilde{\theta}_1(\tilde{Y}_{t+1} - \tilde{c}) - \tilde{\theta}_1^2(\tilde{Y}_{t+2} - \tilde{c}) + \dots + \varepsilon_{t-1} \\
&= \frac{\tilde{c}}{1 + \tilde{\theta}_1} - \sum_{i=1}^{\infty} (-\tilde{\theta}_1)^i \tilde{Y}_{t+i} + \varepsilon_{t-1}
\end{aligned}$$

Y_t depends on all future random variables $Y_{t+1}, Y_{t+2}, \dots$

Invertibility of MA process

Definition

The MA process $\{Y_t, t = 0, 1, ...\}$ is called invertible if its AR(∞) representation is a causal function, i.e. Y_t depends only on Y_s, $s < t$, and doesn't depend on Y_s, $s > t$, for all $t = 0, 1, ...$

For non-invertible representation of MA process it is necessary to know the future values $y_{t+1}, y_{t+2}, ...$ to calculate y_t

MA(1) process with $|\theta_1| < 1$ is invertible:

$$Y_t = \frac{c}{1+\theta_1} - \sum_{i=1}^{\infty} (-\theta_1)^i Y_{t-i} + \varepsilon_t$$

MA(1) process with $|\theta_1| > 1$ is non-invertible:

$$Y_t = \frac{c}{1+\theta_1^{-1}} - \sum_{i=1}^{\infty} (-\theta_1)^{-i} Y_{t+i} + \theta_1 \varepsilon_{t-1}$$

Formal Definition of Invertibility

Definition

The MA(q) process $Y_t = c + \theta(L)\varepsilon_t$, where

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_p L^p$$

is invertible if its inverse moving average operator

$$\pi(L) = \theta^{-1}(L) = 1 - \pi_1 L - \pi_2 L^2 - \ldots$$

has absolutely summable coefficients $\sum_{i=0}^{\infty} |\pi_i| < \infty$.

Absolute summability of $\{\pi_1, \pi_2, \ldots\}$ implies that ε_t will be finite:

$$\varepsilon_t = \pi(L)(Y_t - c) = -\tilde{\mu} + Y_t - \sum_{i=1}^{\infty} \pi_i Y_{t-i} < \infty$$

Invertibility of MA(1) Process

Moving average operator of MA(1) process:

$$\theta(L) = 1 + \theta_1 L$$

Inverse average operator operator:

$$\pi(L) = 1 - \theta_1 L + \theta_1^2 L^2 - \dots$$

Coefficients $\pi_0 = 1,\ \pi_1 = -\theta_1,\ \pi_2 = \theta_1^2, \dots$

- $|\theta_1| < 1$:

$$1 + |\theta_1| + |\theta_1|^2 + \dots = \frac{1}{1 - |\theta_1|} < \infty$$

$\Rightarrow$ MA(1) process is invertible

- $|\theta_1| > 1$:

$$1 + |\theta_1| + |\theta_1|^2 + \dots \to \infty$$

$\Rightarrow$ MA(1) process is non-invertible

Criterion of Invertibility

Theorem

The MA(q) process $Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}$ is invertible iff

$$|z_i| > 1 \quad \forall i = 1, \ldots, q$$

where $z_1, \ldots, z_q$ are roots of its characteristic polynomial

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q$$

MA(q) process is invertible $\Leftrightarrow$ all roots lie outside the unit circle

For MA(1) process:

Characteristic polynomial: $\theta(z) = 1 + \theta_1 z$, its root $z = -\frac{1}{\theta_1}$

Invertibility condition: $\frac{1}{|\theta_1|} > 1 \Rightarrow |\theta_1| < 1$

Factorization of Moving Average Operator

Characteristic polynomial of MA(q) process:

$$\theta(z) = \theta_q \prod_{i=1}^{q} (z - z_i) = \theta_q \prod_{i=1}^{q} z_i \left(\frac{z}{z_i} - 1 \right)$$

$$= \theta_q (-1)^q z_1 \dots z_q \prod_{i=1}^{q} \left(1 - \frac{z}{z_i}\right) = \prod_{i=1}^{q} \left(1 - \frac{z}{z_i}\right)$$

If all roots $|z_i| > 1$, $i = 1, \dots, q$, then the inverse moving average operator can be represented as

$$\pi(L) = \theta^{-1}(L) = \frac{1}{1 + \theta_1 L + \dots + \theta_q L^q} = \frac{1}{\prod_{i=1}^{q} \left(1 - \frac{1}{z_i} L\right)}$$

$$= \prod_{i=1}^{q} \left(1 + \frac{1}{z_i} L + \frac{1}{z_i^2} L^2 + \dots\right) = 1 - \pi_1 L - \pi_2 L^2 - \dots$$

AR(∞) Form of MA(q) Process

The coefficients $\pi_1, \pi_2, ...$ can be obtained by imposing the cancellation of powers of L in identity equation $\theta(L)\pi(L) = 1$:

$$(1 + \theta_1 L + \theta_2 L^2 + ...)(1 - \pi_1 L - ... - \pi_p L^p) = 1$$

$$\theta_1 = \pi_1$$

$$\theta_2 = \pi_1\theta_1 + \pi_2$$

$$\theta_k = \pi_1\theta_{k-1} + \pi_2\theta_{k-2} + ... + \pi_k = \sum_{i=1}^{k} \pi_i\theta_{k-i} \quad (\theta_0 = 1)$$

MA(q) process in operator form:

$$Y_t = c + \theta(L)\varepsilon_t$$

MA(q) process in AR(∞) form:

$$\varepsilon_t = (1 - \pi_1 L - \pi_2 L^2 + ...)(Y_t - c) = -\tilde{\mu} + Y_t - \sum_{i=1}^{\infty} \pi_i Y_{t-i}$$

Ambiguity of MA(q) Processes

Consider two MA(1) processes:

$$Y_t = \varepsilon_t + \frac{1}{5}\varepsilon_{t-1}, \quad \varepsilon_t \sim N(0, 25), t \in \mathbb{Z}$$

$$Z_t = \varepsilon_t + 5\varepsilon_{t-1}, \quad \varepsilon_t \sim N(0, 1), t \in \mathbb{Z}$$

These processes have different representations but they are indistinguishable:

$$\mathrm{M}[Y_t] = \mathrm{M}[Z_t] = 0$$

$$c_Y(\tau) = c_Z(\tau) = \begin{cases} 26, & \tau = 0, \\ 5, & \tau = 1, \\ 0, & \tau > 1, \end{cases}$$

It can be shown that for MA(q) process always exists invertible representation. It can be obtained by inverting the roots of characteristic polynomial that are smaller than 1

It's preferable to choose invertible representation

AR(p) Process. Summary

Model	$Y_t = c + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \varepsilon_t$
Characteristic polynomial	$\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$
Operator form	$\phi(L) Y_t = c + \varepsilon_t$
Inverse operator	$\psi(L) = \phi^{-1}(L) = 1 + \psi_1 L + \psi_2 L^2 + ...,$ $\psi_k = \phi_1 \psi_{k-1} + ... + \phi_p \psi_{k-p}, \quad k = 1, 2, ...,$ $\psi_0 = 1, \quad \psi_k = 0, \quad k < 0$
MA(∞) form	$Y_t = \phi^{-1}(L)(c + \varepsilon_t)$
Stationarity	roots $\lvert z_i \rvert > 1 \quad \forall i = 1, ..., p$
Causality	roots $\lvert z_i \rvert > 1 \quad \forall i = 1, ..., p$
Inversibility	Always
Ergodicity	roots $\lvert z_i \rvert > 1 \quad \forall i = 1, ..., p$
Memory	Infinite, remembers all previous innovations

MA(q) Process. Summary

Model	$Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$
Characteristic polynomial	$\theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$
Operator form	$Y_t = c + \theta(L)\varepsilon_t$
Inverse operator	$\pi(L) = \theta^{-1}(L) = 1 - \pi_1 L - \pi_2 L^2 - ...,$ $\theta_k = \pi_1 \theta_{k-1} + ... + \pi_q \theta_{k-q}, \quad k = 1, 2, ...,$ $\theta_0 = 1, \quad \theta_k = 0, \quad k < 0$
AR(∞) form	$\varepsilon_t = \theta^{-1}(L)(Y_t - c)$
Stationarity	Always
Causality	Always
Inversibility	roots $\lvert z_i \rvert > 1 \quad \forall i = 1, ..., q$
Ergodicity	Always
Memory	Finite, remembers only q previous innovations

Wold's Theorem

The AR and MA processes are specific cases of a general representation of stationary processes obtained by Wold

Theorem (Wold, 1938)

Let $\{Y_t, t = 0, 1, ...\}$ be WSS process with finite mean μ that does not contain deterministic component. Then it can be written as a linear function of zero-mean uncorrelated random variables $\{\varepsilon_t, t \in \mathbb{Z}\}$:

$$Y_t = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$$

where $\mathrm{M}[\varepsilon_t] = 0$, $\mathrm{D}[\varepsilon_t] = \sigma^2$, $cov(\varepsilon_t, \varepsilon_{t-\tau}) = 0$ for all $t \geq 0$, $\tau \geq 1$

If the process $\{Y_t, t = 0, 1, ...\}$ contains deterministic component (trend) it should be extracted firstly

Wold's Theorem. Notes

- Wold's theorem guarantees that any WSS process has a linear representation
- The random variables $\{\varepsilon_t, t \in \mathbb{Z}\}$ actually is a white noise process, thus Wold's theorem guarantees that any WSS process can be represented in MA(∞) form, which is called as general linear representation of WSS process
- If the process $\{Y_t, t = 0, 1, ...\}$ can be represented as a function of Gaussian white noise $\{\varepsilon_t, t = 0, 1, ...\}$, it will be Gaussian process and the weak coincides with strict stationarity
- Usually $\psi_0 = 1$ assumed
- If the coefficients ψ_i are all zeros after lag q then the general linear representation reduces to MA(q) form:

$$Y_t = \mu + \sum_{i=0}^{q} \psi_i \varepsilon_{t-i}$$

Expectation and Autocovariance of WSS Process

Expectation:

$$\mathrm{M}[Y_t] = \left[\mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}\right] = \mu$$

Variance:

$$\mathrm{D}[Y_t] = \mathrm{D}\left[\mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}\right] = \sigma^2 \sum_{i=0}^{\infty} \psi_i^2$$

The process $\{Y_t, t = 0, 1, \ldots\}$ will have finite variance if $\sum\limits_{i=0}^{\infty} \psi_i^2 < \infty$

Autocovariance:

$$Y_t = \mu + \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \ldots + \psi_\tau \varepsilon_{t-\tau} + \ldots$$

$$Y_{t-\tau} = \mu + \psi_0 \varepsilon_{t-\tau} + \psi_1 \varepsilon_{t-1-\tau} + \psi_2 \varepsilon_{t-2-\tau} + \ldots$$

$$c(\tau) = \mathrm{M}[(Y_t - \mu)(Y_{t-\tau} - \mu)] = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+\tau}$$

AR(∞) and MA(∞) Representations in Operator Form

Wold's theorem gives MA(∞) representation of WSS process $\{Y_t\}$:

$$Y_t = \mu + \psi(L)\varepsilon_t$$

where

$$\psi(L) = 1 + \psi_1 L + \psi_2 L^2 + ...$$

Inverse representation is AR(∞):

$$\psi^{-1}(L)(Y_t - \mu) = \varepsilon_t$$

where

$$\psi^{-1}(L) = \pi(L) = 1 - \pi_1 L - \pi_2 L^2 - ...$$

$$(1 - \pi_1 L - \pi_2 L^2 - ...)(Y_t - \mu) = \varepsilon_t$$

$$(Y_t - \mu) - (\pi_1 L + \pi_2 L^2 + ...)(Y_t - \mu) = \varepsilon_t$$

$$Y_t = \mu + (\pi_1 L + \pi_2 L^2 + ...)(Y_t - \mu) + \varepsilon_t$$

Coefficients of AR(∞) and MA(∞) Representations

The coefficients $\pi_1, \pi_2, \dots$ can be obtained by imposing the cancellation of powers of L in identity equation $\psi(L)\pi(L) = 1$:

$$(1 + \psi_1 L + \psi_2 L^2 + \dots)(1 - \pi_1 L - \dots - \pi_p L^p) = 1$$

$$\psi_1 = \pi_1$$

$$\psi_2 = \pi_1 \psi_1 + \pi_2$$

$$\psi_k = \pi_1 \psi_{k-1} + \pi_2 \psi_{k-2} + \dots + \pi_k = \sum_{i=1}^{k} \pi_i \psi_{k-i} \quad (\psi_0 = 1)$$

WSS process $\{Y_t\}$ in MA(∞) form: $Y_t = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$

WSS process $\{Y_t\}$ in AR(∞) form: $Y_t = \tilde{\mu} + \sum_{i=1}^{\infty} \pi_i Y_{t-i} + \varepsilon_t$

MA(∞) Representation of Stationary AR(p) Process

Note that the lag operator $\phi(L)$ of stationary AR(p) process has finite number of terms:

$$\phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$$

but the inverse operator $\psi(L)$ has infinite, i.e. it's MA(∞) process:

$$\psi(L) = \phi^{-1}(L) = 1 + \psi_1 L + \psi_2 L^2 + \ldots$$

For example, AR(1) process

$$Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t, \quad |\phi_1| < 1$$

has MA(∞) representation:

$$Y_t = \mu + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i} = \mu + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \ldots$$

Here $\psi_0 = 1$, $\psi_1 = \phi_1$, $\psi_2 = \phi_1^2$, ... is infinite sequence but it is characterized by only one parameter ϕ_1

AR(p), MA(q) and General Linear Process

The AR(p) processes can be considered as particular case of the general linear process characterized by the facts that:

- all MA coefficients ψ_i are different from zero
- there are restrictions on the ψ_i, that depend on the order of the process
- coefficients ψ_i satisfy the sequence

$$\psi_k = \phi_1\psi_{k-1} + \phi_2\psi_{k-2} + \ldots + \phi_k = \sum_{i=1}^{k} \phi_i\psi_{k-i} \quad (\psi_0 = 1)$$

  with initial conditions that depend on the order of the process

The MA(q) processes can be considered as particular case of the general linear process characterized by the fact that:

- all MA coefficients ψ_i are zeros after lag q
- MA coefficients ψ_i up to lag q can be arbitrary

Autoregressive Moving Average Process

Could we combine these properties and allow us to represent the processes whose first q coefficients can be arbitrary, whereas the following ones decay according to simple rule?

Definition

Discrete-time autoregressive moving average process of AR order p and MA order q (ARMA(p,q) process) $\{Y_t, t = 0, 1, ...\}$ is

$$Y_t = c + \phi_1 Y_{t-1} + ...\phi_p Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}, \quad t \geq p$$

where $\{\varepsilon_t, t = -q, -q+1, ...\}$ is a discrete-time white noise and c, $\phi_1, ..., \phi_p$, $\theta_1, ..., \theta_q$ are constants

Y_t depends only on current and previous innovations $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...$ and for any $s > t$ the Y_t and ε_s are independent

AR, MA and ARMA Processes

AR and MA processes are particular cases of ARMA process:

$$AR(p) = ARMA(p, 0)$$

$$MA(q) = ARMA(0, q)$$

The AR part of ARMA process implies the MA(∞) structure and imposes restrictions on the coefficients ψ_i

The MA part of ARMA process modifies first q coefficients ψ_i in MA(∞) representation and makes them arbitrary

In practice it's convenient to use a few number of parameters to model the real time series. If parameters in MA(∞) representation have some structure, the AR part should be included to the model

Poles and Zeros of ARMA Process

ARMA(p,q) process in operator form:

$$(1 - \phi_1 L - \dots - \phi_p L^p) Y_t = c + (1 + \theta_1 L + \dots + \theta_q L^q)\varepsilon_t$$

$$\phi(L) Y_t = c + \theta(L)\varepsilon_t$$

where

$$\phi(L) = 1 - \phi_1 L - \dots - \phi_p L^p$$

$$\theta(L) = 1 + \theta_1 L + \dots + \theta_q L^q$$

The lag operator polynomials can be factorized:

$$\phi(L) = \prod_{i=1}^{p} \left(1 - \frac{1}{p_i} L\right), \quad \theta(L) = \prod_{i=1}^{q} \left(1 - \frac{1}{z_i} L\right)$$

Roots $p_1, \dots, p_p$ of $\phi(z)$ are called as poles of ARMA process

Roots $z_1, \dots, z_q$ of $\theta(z)$ are called as zeros of ARMA process

ARMA Parameters Redundancy

If $Y_t = \varepsilon_t$ then $\xi(L)Y_t = \xi(L)\varepsilon_t$, where $\xi(L)$ is some lag operator polynomial

Example 1:

$$\psi(L) = 1 - 0.5L \quad \Rightarrow \quad Y_t - 0.5Y_{t-1} = \varepsilon_t - 0.5\varepsilon_{t-1}$$

It looks like ARMA(1,1) but in fact, it is $AR(0)$ process $Y_t = \varepsilon_t$

Example 2:

$$Y_t = 0.4Y_{t-1} + 0.45Y_{t-2} + \varepsilon_t + \varepsilon_{t-1} + 0.25\varepsilon_{t-2}$$

$$\phi(z) = 1 - 0.4z - 0.45z^2 = (1 + 0.5z)(1 - 0.9z)$$

$$\theta(z) = 1 + z + 0.25z^2 = (1 + 0.5z)^2$$

Removing the common term $(1 + 0.5z)$ gives the reduced ARMA(1,1) process

$$Y_t = 0.9Y_{t-1} + \varepsilon_t + 0.5\varepsilon_{t-1}$$

AR(∞) and MA(∞) Representations in Operator Form

MA(∞) representation (if inverse AR operator $\phi^{-1}(L)$ exists):

$$Y_t = \phi^{-1}(L)(c + \theta(L)\varepsilon_t) = \mu + \frac{\theta(L)}{\phi(L)}\varepsilon_t = \mu + \psi(L)\varepsilon_t$$

where

$$\mu = \frac{c}{1 - \phi_1 - ... - \phi_p}$$

$$\psi(L) = \frac{\theta(L)}{\phi(L)} = 1 + \psi_1 L + \psi_2 L^2 + ...$$

AR(∞) representation (if inverse MA operator $\theta^{-1}(L)$ exists):

$$\varepsilon_t = \frac{\phi(L)}{\theta(L)}(Y_t - c) = \pi(L)(Y_t - c)$$

where

$$\pi(L) = \frac{\phi(L)}{\theta(L)} = 1 - \pi_1 L - \pi_2 L^2 - ...$$

Stationarity and Invertibility of ARMA Process

Assume that $\phi(L)$ and $\theta(L)$ have no common factors

ARMA(p,q) as MA(∞) process

MA(∞) representation (ARMA process as a sum of stationary white noise process) exists if $\phi^{-1}(L)$ exists

ARMA(p,q) process is stationary if its AR part is stationary, i.e. all roots of $\phi(L)$ are outside of unit circle

ARMA(p,q) as AR(∞) process

AR(∞) representation (white noise process as a sum of ARMA process) exists if $\theta^{-1}(L)$ exists

ARMA(p,q) process is invertible if its MA part is invertible, i.e. all roots of $\theta(L)$ are outside of unit circle

Poles correspond to stationarity, zeros correspond to invertibility

Expectation of ARMA Process

ARMA(p,q) process:

$$Y_t = c + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$$

For stationary ARMA process:

$$\mathrm{M}[Y_t] = c + \phi_1 \mathrm{M}[Y_{t-1}] + ... + \phi_p \mathrm{M}[Y_{t-p}]$$

$$\mu = \mathrm{M}[Y_t] = \frac{c}{1 - \phi_1 - ... - \phi_p}$$

Replacing c with $\mu(1 - \phi_1 - ... - \phi_p)$ the process can be written w.r.t. deviations from the mean:

$$Y_t = \mu(1 - \phi_1 - ... - \phi_p) + \phi_1 Y_{t-1} + ... \phi_p Y_{t-p} + \varepsilon_t + ... + \theta_q \varepsilon_{t-q}$$

$$Y_t - \mu = \phi_1 (Y_{t-1} - \mu) + ... + \phi_p (Y_{t-p} - \mu) + \varepsilon_t + ... + \theta_q \varepsilon_{t-q}$$

$$\tilde{Y}_t = \phi_1 \tilde{Y}_{t-1} + ... \phi_1 \tilde{Y}_{t-1} + \varepsilon_t + ... + \theta_q \varepsilon_{t-q}$$

where $\tilde{Y}_t = Y_t - \mu$ is a centered random variable $Y_t, t = 0, 1, ...$

Autocovariance of ARMA Process

Centered ARMA(p,q) process:

$$\tilde{Y}_t = \phi_1 \tilde{Y}_{t-1} + ... + \phi_p \tilde{Y}_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$$

To calculate the autocovariances we multiply ARMA process by $\tilde{Y}_{t-\tau}$ and take expectations (Yule-Walker's method):

$$\begin{aligned} c(\tau) = &\phi_1 c(\tau - 1) + ... + \phi_p c(\tau - p) \\ &+ \mathrm{M}[\tilde{Y}_{t-\tau}\varepsilon_t] + \theta_1 \mathrm{M}[\tilde{Y}_{t-\tau}\varepsilon_{t-1}] + ... + \theta_q \mathrm{M}[\tilde{Y}_{t-\tau}\varepsilon_{t-q}] \end{aligned}$$

For $\tau > q$ all the terms $\theta_i \mathrm{M}[\tilde{Y}_{t-\tau}\varepsilon_{t-i}]$ are cancelled:

$$c(\tau) - \phi_1 c(\tau - 1) - ... - \phi_p c(\tau - p) = 0$$

In operator form:

$$\phi(L) c(\tau) = 0, \quad \tau > q$$

$\Rightarrow$ autocovariances for $\tau > q$ follow a decay dictated by the autoregressive part $\phi(L)$

ARMA(1,1) Process

ARMA(1,1) process:

$$Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

In operator form:

$$(1 - \phi_1 L) Y_t = c + (1 + \theta_1 L)\varepsilon_t$$

AR operator: $\phi(L) = 1 - \phi_1 L$, **pole:** $z = \frac{1}{\phi_1}$
MA operator: $\theta(L) = 1 + \theta_1 L$, **zero:** $z = -\frac{1}{\theta_1}$
$|\phi_1| < 1 \Rightarrow \{Y_t\}$ is stationary, $|\theta_1| < 1 \Rightarrow \{Y_t\}$ is invertible
Expectation:

$$\mu = \frac{c}{1 - \phi_1}$$

Centered process:

$$\tilde{Y}(t) = \phi_1 \tilde{Y}_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

where $\tilde{Y}_t = Y_t - \mu$

Autocovariance of ARMA(1,1) Process

To calculate the autocovariances we multiply $\tilde{Y}_t$ process by $\tilde{Y}_{t-\tau}$ and take expectations:

$$\tilde{Y}_t = \phi_1 \tilde{Y}_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

$$\mathrm{M}[\tilde{Y}_t \tilde{Y}_{t-\tau}] = \phi_1 \mathrm{M}[\tilde{Y}_{t-1} \tilde{Y}_{t-\tau}] + \mathrm{M}[\varepsilon_t \tilde{Y}_{t-\tau}] + \theta_1 \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-\tau}]$$

$$c(\tau) = \phi_1 c(\tau - 1) + \mathrm{M}[\varepsilon_t \tilde{Y}_{t-\tau}] + \theta_1 \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-\tau}]$$

For $\tau = 0$:

$$\mathrm{M}[\varepsilon_t \tilde{Y}_{t-\tau}] = \mathrm{M}[\varepsilon_t \tilde{Y}_t] = \mathrm{M}[\varepsilon_t(\phi_1 \tilde{Y}_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1})] = \mathrm{M}[\varepsilon_t^2] = \sigma^2$$

$$\begin{aligned}\mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-\tau}] &= \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_t] = \mathrm{M}[\varepsilon_{t-1}(\phi_1 \tilde{Y}_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1})] \\ &= \phi_1 \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-1}] + \theta_1 \mathrm{M}[\varepsilon_{t-1}^2] = \phi_1 \sigma^2 + \theta_1 \sigma^2 \\ &= (\phi_1 + \theta_1)\sigma^2\end{aligned}$$

$$c(0) = \phi_1 c(-1) + \sigma^2 + \theta_1(\phi_1 + \theta_1)\sigma^2 = \phi_1 c(1) + (1 + \phi_1\theta_1 + \theta_1^2)\sigma^2$$

Autocovariance of ARMA(1,1) Process

For $\tau = 1$:

$$\mathrm{M}[\varepsilon_t \tilde{Y}_{t-\tau}] = \mathrm{M}[\varepsilon_t \tilde{Y}_{t-1}] = 0$$

$$\mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-\tau}] = \mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-1}] = \sigma^2$$

$$c(1) = \phi_1 c(0) + \theta_1 \sigma^2$$

Substituting $c(1)$ in the expression for $c(0)$:

$$c(0) = \phi_1 c(1) + (1 + \phi_1\theta_1 + \theta_1^2)\sigma^2 = \phi_1(\phi_1 c(0) + \theta_1\sigma^2) + (1 + \phi_1\theta_1 + \theta_1^2)\sigma^2$$

we obtain

$$c(0) = \frac{1 + 2\phi_1\theta_1 + \theta_1^2}{1 - \phi_1^2}\sigma^2$$

$$c(1) = \phi_1 c(0) + \theta_1\sigma^2 = \frac{(1 + \phi_1\theta_1)(\phi_1 + \theta_1)}{1 - \phi_1^2}\sigma^2$$

Autocovariance of ARMA(1,1) Process

For $\tau > 1$:

$$\mathrm{M}[\varepsilon_t \tilde{Y}_{t-\tau}] = 0$$

$$\mathrm{M}[\varepsilon_{t-1} \tilde{Y}_{t-\tau}] = 0$$

$$c(\tau) = \phi_1 c(\tau - 1)$$

The autocovariances $c(\tau)$ for $\tau = 0, ..., q$ of ARMA(p,q) process can have a complex structure determined by the MA and AR parts

For $\tau > q$ the autocovariances depend only on previous autocovariances. In general case it's a mixture of exponentials and sinusoids, determined exclusively by the AR part

If $\phi_1 = -\theta_1$: $\phi(L)$ and $\theta(L)$ have common root, thus the ARMA(1,1) process reduces to AR(0) process:

$$(1 - \phi_1 L)\tilde{Y}_t = (1 - \phi_1 L)\varepsilon_t$$

$$\tilde{Y}_t = \varepsilon_t, \quad c(0) = \sigma^2, \quad c(\tau) = 0 \quad \forall \tau > 0$$

Sum of AR Processes

One reason that explains why the ARMA processes are frequently found in practice is that summing AR processes results in an ARMA process

For example, let's mix independent AR(1) and AR(0) processes:

$$Y_t = \phi_1 Y_{t-1} + \varepsilon_t \quad \text{(AR(1) process)}$$

$$X_t = \nu_t \quad \text{(AR(0) process)}$$

The resulting process $Z_t = Y_t + X_t$ can be interpreted as the result of observing an AR(1) process $\{Y_t\}$ with a certain measurement error $\{X_t\}$

Expectation:

$$\mathrm{M}[Z_t] = \mathrm{M}[Y_t + X_t] = \mathrm{M}[Y_t] = 0$$

Sum of AR Processes and ARMA Process

Variance:

$$c_Z(0) = \mathrm{D}[Z_t] = \mathrm{D}[Y_t + X_t] = \mathrm{D}[Y_t] + \sigma_\nu^2 = c_Y(0) + \sigma_\nu^2$$

Autocovariance:

$$c_Z(\tau) = \mathrm{M}[Z_t Z_{t-\tau}] = \mathrm{M}[(Y_t + X_t)(Y_{t-\tau} + X_{t-\tau})] = c_Y(\tau) = \phi_1^\tau c_Y(0)$$

Substituting $c_Y(0)$ from the expression for variance:

$$c_Z(\tau) = \phi_1^\tau (c_Z(0) - \sigma_\nu^2) = \phi_1 c_Z(\tau - 1)$$

For $\tau = 1$: $c_Z(1) = \phi_1 c_Z(0) - \phi_1 \sigma_\nu^2$

It was shown that autocovariance $c(\tau)$ of ARMA(1,1) process:

$$c(1) = \phi_1 c(0) + \theta_1 \sigma^2, \quad c(\tau) = \phi_1 c(\tau - 1), \quad \tau > 1$$

So we conclude that process $\{Z_t\}$ follows ARMA(1,1) model with AR parameter equal to ϕ_1 and MA parameter is determined by ϕ_1 and variances σ^2 and σ_ν^2

Sum of Stationary Processes and ARMA Process

It can be shown that:

- any sum of independent AR processes is ARMA process:

$$AR(p) + AR(q) = ARMA(p + q, \max(p, q))$$

  particularly:

$$AR(p) + AR(0) = ARMA(p, p)$$

- any sum of independent MA processes is MA process:

$$MA(q_1) + MA(q_2) = MA(\max(q_1, q_2))$$

- any sum of independent ARMA processes is ARMA process:

$$ARMA(p_1, q_1) + ARMA(p_2, q_2) = ARMA(a, b)$$

  where $a \leq p_1 + p_2$, $b \leq \max(p_1 + q_1, p_2 + q_2)$

Sum of Stationary Processes and ARMA Process. Notes

- Whenever we observe processes that are the sum of others, and some of them have an AR structure, we expect to observe ARMA processes
- Under certain conditions any sum of stochastic processes tends to be ARMA process. It justifies the popularity of ARMA models for time series modelling
- In practice many real series are approximated well by means of AR or MA processes. It is due to cancellation of similar roots of AR and MA characteristic polynomials of ARMA model
- If it is known that a real world process is stationary ARMA process then it is ergodic. Just one realization is needed to estimate its mean and autocovariance function