Стационарные и эргодические случайные процессы

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Random Process

Definition

A random (or stochastic) process X is a collection of random variables $\{X(t)\}_{t\in\mathscr{T}}$ that is indexed by some mathematical set $\mathscr{T}$

Each random variable $X(t)$ of the stochastic process X is uniquely associated with an element $t \in \mathscr{T}$, where $\mathscr{T}$ is the index set

Notations:

$$X, \quad \{X(t)\}_{t\in\mathscr{T}}, \quad \{X_t\}_{t\in\mathscr{T}}, \quad \{X(t), t\in\mathscr{T}\}$$

$\mathscr{T}$ continuous $\Rightarrow$ X is called a continuous-time random process
$\mathscr{T}$ discrete $\Rightarrow$ X is called a discrete-time random process
$\mathscr{T}$ integer $\Rightarrow$ X is called a random sequence (time series)

The term random function is also used to refer to a random (stochastic) process

Ensemble of Signals and Sample Function

A random process is the process which generates an ensemble of random signals

Ensemble is a collection (a family) of all possible random signals generated by a random process

Sample function (realization, sample path, trajectory of random process) is one specific random signal $\{x(t), t \in \mathscr{T}\}$, generated by a random process X

The value of a sample function $x(t)$ at time t is called the amplitude of random signal at time $t \in \mathscr{T}$

The amplitude $x(t)$ at fixed time t, is a realization of corresponding random variable $X(t)$, $t \in \mathscr{T}$

Index Set, State Space and Sample Space

Index set $\mathscr{T}$ is a space of all possible values of independent variable (time) t

Examples of index sets: $\mathscr{T} = \mathbb{R}, \quad \mathscr{T} = \mathbb{Z}, \quad \mathscr{T} = \{1, 2, ...\}$

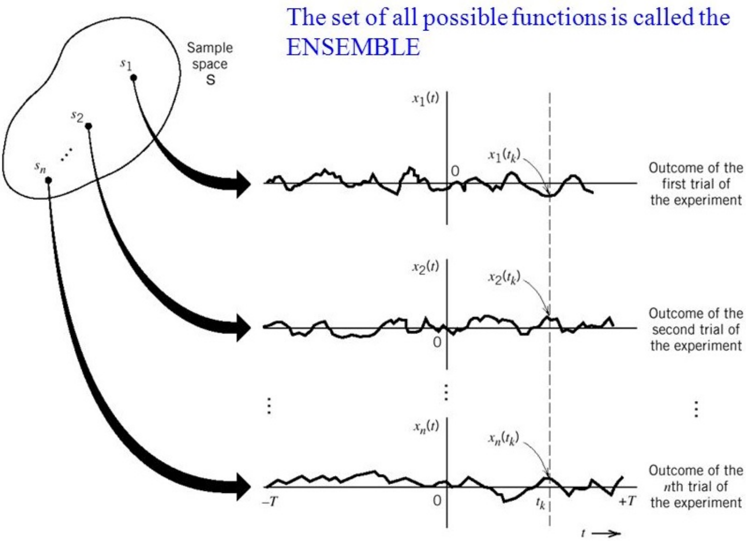
State space $\mathscr{X}$ is a mathematical space from which each random variable $X(t)$, $t \in \mathscr{T}$, of the process X takes values (can be continuous or discrete)

For any sample function: $x(t) \in \mathscr{X}$, $t \in \mathscr{T}$

Sample space Ω is a set of all possible outcomes of the underlying random experiment. Each outcome $\omega \in \Omega$ leads to a corresponding sample function $x(t)$, $t \in \mathscr{T}$

A random process X provides a mathematical model for an ensemble of random signals and represents mapping of the sample space Ω into a set of signals

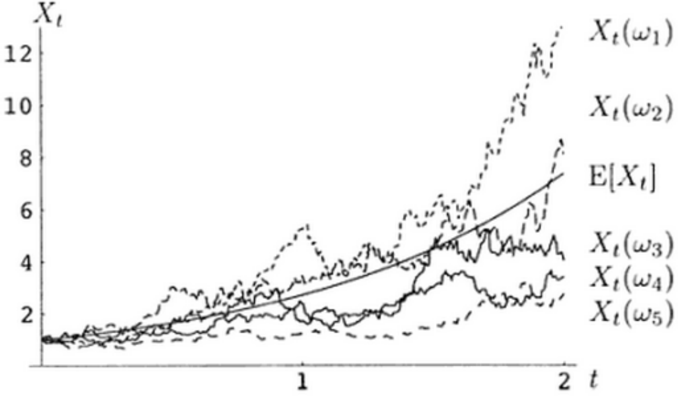
Random Process. Illustration 1



Random Process. Illustration 2

A sample function is formed by taking a single possible value of each random variable $X(t)$, $t \in \mathscr{T}$, of the stochastic process X

Sample functions of a random process



Univariate CDF and PDF

Stochastic process can be viewed as a function of two variables – time t and outcome $\omega \in \Omega$, where Ω is a sample space

For a given time $t \in \mathscr{T}$ we get a random variable $X(t)$, $t = const$
For a given outcome $\omega \in \Omega$ we get a function of time $x(t)$, $t \in \mathscr{T}$

The univariate cumulative distribution function (CDF) of a real-valued random signal X is defined as

$$F(x,t) = F_{X(t)}(x) = P(X(t) < x), \quad t \in \mathscr{T}$$

The univariate probability density function (PDF) of a continuous-amplitude real-valued random signal X is defined as the derivative of the univariate CDF:

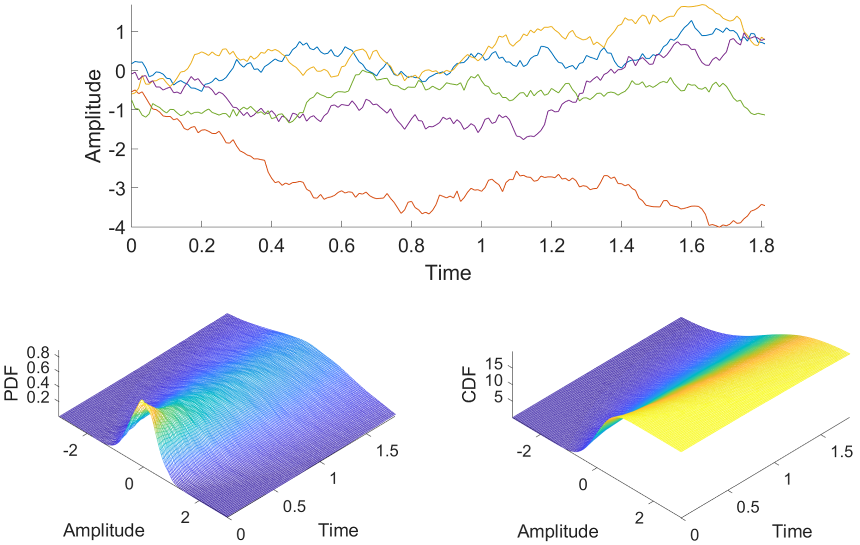
$$f(x,t) = \frac{\partial F(x,t)}{\partial x}, \quad t \in \mathscr{T}$$

Properties of Univariate CDF and PDF

Univariate CDF and PDF at any given time $t \in \mathscr{T}$ characterize the random variable $X(t)$

- $0 \le F(x,t) \le 1$, for all $x \in (-\infty, +\infty)$
- $F(-\infty, t) = 0$
- $F(+\infty, t) = 1$
- $F(x_1, t) \le F(x_2, t), \quad x_1 \le x_2$
- $P(a \le X(t) \le b) = F(b,t) - F(a,t), \quad a \le b$
- $\int\limits_{-\infty}^{+\infty} f(x,t)dx = 1$
- $F(x,t) = \int\limits_{-\infty}^{x} f(\xi,t)d\xi$
- $f(-\infty, t) = f(+\infty, t) = 0$
- $P(a \le X(t) \le b) = \int\limits_{a}^{b} f(x,t)dx$

Univariate CDF and PDF. Illustrations



Expectation and Variance

Expectation of time series X is a deterministic function $m(t)$, $t \in \mathscr{T}$:

$$m(t) = \mathrm{M}[X(t)] = \int\limits_{-\infty}^{+\infty} x f(x,t) dx$$

Variance of time series X is a deterministic function $d(t)$, $t \in \mathscr{T}$:

$$\sigma^2(t) = \mathrm{D}[X(t)] = \int\limits_{-\infty}^{+\infty} (x - m(t))^2 f(x,t) dx$$

Let's $Y(t) = \varphi(X(t))$, $t \in \mathscr{T}$, where $\varphi(\cdot)$ is an arbitrary function

Expectation of time series Y at time $t \in \mathscr{T}$ is defined as:

$$\mathrm{M}[Y(t)] = \mathrm{M}[\varphi(X(t))] = \int\limits_{-\infty}^{+\infty} \varphi(x) f(x,t) dx$$

Estimations by Ensemble Averaging

Expectation $m(t)$ and variance $\sigma^2(t)$ at time $t \in \mathscr{T}$ can be estimated by ensemble averaging:

$$\hat{m}_N(t) = \frac{1}{N} \sum_{n=1}^{N} x_n(t)$$

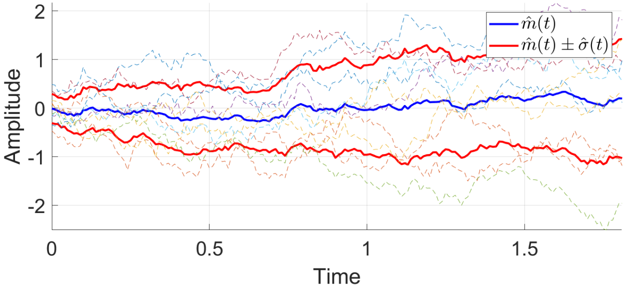
$$\hat{\sigma}_N^2(t) = \frac{1}{N} \sum_{n=1}^{N} (x_n(t) - \hat{m}_N(t))^2$$

where $x_n(t)$ is a value of random process X in n-th realization at time t, $t \in \mathscr{T}$; N is a number of realizations used for estimation

It can be shown that $\hat{m}_N(t)$ and $\hat{\sigma}_N^2(t)$ are consistent estimators of expectation $\mathrm{M}[X(t)]$ and variance $\mathrm{D}[X(t)]$:

$$\hat{m}_N(t) \to m(t), \quad \hat{\sigma}_N^2(t) \to \sigma^2(t) \quad \text{when} \quad N \to \infty$$

Ensemble Averages. Illustration



Standardization of time series X:

$$Y(t) = \frac{X(t) - m(t)}{\sigma(t)}, \quad t \in \mathscr{T}$$

Properties of standardized time series Y:

$$\mathrm{M}[Y(t)] = 0, \quad \mathrm{D}[Y(t)] = 1, \quad \forall t \in \mathscr{T}$$

Bivariate CDF and PDF

Choose some pair of time moments $t_1 \in \mathscr{T}$ and $t_2 \in \mathscr{T}$

For a pair of corresponding random variables $X(t_1)$ and $X(t_2)$ we can specify their joint probability distribution functions

The bivariate (or joint) CDF of a random signal X is defined as

$$\begin{aligned} F(x_1, x_2; t_1, t_2) &= F_{X(t_1)X(t_2)}(x_1, x_2) \\ &= P(X(t_1) < x_1 \& X(t_2) < x_2), \quad t_1 \in \mathscr{T},\ t_2 \in \mathscr{T} \end{aligned}$$

The bivariate (or joint) PDF of a random signal X is defined as the second derivative of the bivariate CDF:

$$f(x_1, x_2; t_1, t_2) = \frac{\partial^2 F(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}, \quad t_1 \in \mathscr{T},\ t_2 \in \mathscr{T}$$

Properties of Bivariate CDF and PDF

Bivariate CDF and PDF at any given times $t_1 \in \mathscr{T}$, $t_2 \in \mathscr{T}$ characterize the random vector $(X(t_1), X(t_2))$

- $F(-\infty, x_2; t_1, t_2) = 0$
- $F(\infty, x_2; t_1, t_2) = F(x_2, t_2)$
- $F(\infty, \infty; t_1, t_2) = 1$
- $f(-\infty, x_2; t_1, t_2) = 0$
- $f(\infty, x_2; t_1, t_2) = 0$
- $f(x_1, t_1) = \int\limits_{-\infty}^{+\infty} f(x_1, x_2; t_1, t_2) dx_2$
- $P(a_1 \leq X(t_1) \leq b_1 \& a_2 \leq X(t_2) \leq b_2) = F(b_1, b_2; t_1, t_2) - F(b_1, a_2; t_1, t_2) - F(a_1, b_2; t_1, t_2) + F(a_1, a_2; t_1, t_2)$

Conditional CDF and PDF

Conditional CDF at time $t_2 \in \mathscr{T}$ given that the process X had value x_1 at time $t_1 \in \mathscr{T}$ is defined as

$$F(x_2, t_2 | x_1, t_1) = P(X(t_2) < x_2 | X(t_1) = x_1)$$

Conditional PDF is the derivative of conditional CDF:

$$f(x_2, t_2 | x_1, t_1) = \frac{\partial F(x_2, t_2 | x_1, t_1)}{\partial x_2}, \quad t_1 \in \mathscr{T},\ t_2 \in \mathscr{T}$$

$$f(x_2, t_2 | x_1, t_1) = \frac{f(x_1, x_2; t_1, t_2)}{f(x_1, t_1)}$$

$$f(x_1, x_2; t_1, t_2) = f(x_2, t_2 | x_1, t_1) f(x_1, t_1)$$

Conditional distributions are used in time series prediction tasks

Joint CDF and PDF

The realization of two-dimensional process (X, Y) is a two-dimensional trajectory $(x(t_1), y(t_1)), (x(t_2), y(t_2)), \dots$

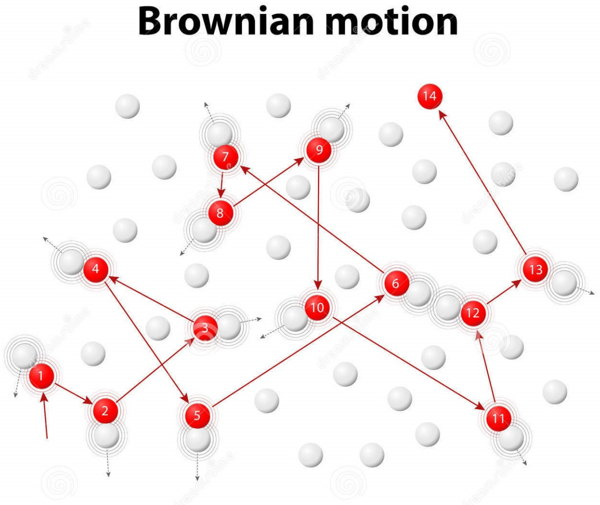
The joint CDF of two-dimensional process (X, Y) is defined as

$$\begin{aligned} F_{XY}(x, y; t_1, t_2) &= F_{X(t_1)Y(t_2)}(x, y) \\ &= P(X(t_1) < x \& Y(t_2) < y), \quad t_1 \in \mathscr{T},\ t_2 \in \mathscr{T} \end{aligned}$$

The joint PDF of two-dimensional process (X, Y) is defined as the second derivative of the joint CDF:

$$f_{XY}(x, y; t_1, t_2) = \frac{\partial^2 F_{XY}(x, y; t_1, t_2)}{\partial x \partial y}, \quad t_1 \in \mathscr{T},\ t_2 \in \mathscr{T}$$

2-D Time Series. Illustration



Autocovariance and Autocorrelation

Autocovariance of time series X is a deterministic function $c(t_1, t_2)$, $t_1 \in \mathscr{T}$, $t_2 \in \mathscr{T}$, defined as

$$c(t_1, t_2) = cov(X(t_1), X(t_2)) = \mathrm{M}[(X(t_1) - m(t_1))(X(t_2) - m(t_2))]$$

$$= \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} (x(t_1) - m(t_1))(x(t_2) - m(t_2)) f(x_1, x_2; t_1, t_2) dx_1 dx_2$$

Autocorrelation of time series X is the autocovariance of standardized time series X:

$$\rho(t_1, t_2) = cov\left(\frac{X(t_1) - m(t_1)}{\sigma(t_1)}, \frac{X(t_2) - m(t_2)}{\sigma(t_2)}\right) = \frac{c(t_1, t_2)}{\sigma(t_1)\sigma(t_2)}$$

Properties of Autocovariance and Autocorrelation

Autocovariance $c(t_1, t_2)$ is the covariance between random variables $X(t_1)$ and $X(t_2)$, $t_1 \in \mathscr{T}$, $t_2 \in \mathscr{T}$

- $c(t_1, t_2) = c(t_2, t_1)$
- $c(t, t) = \sigma^2(t)$
- $|c(t_1, t_2)| \leq \sigma(t_1)\sigma(t_2) = \sqrt{c(t_1, t_1)c(t_2, t_2)}$
- $X(t_1)$ and $X(t_2)$ are independent $\Rightarrow c(t_1, t_2) = 0$
- $c(t_1, t_2) = 0 \Rightarrow X(t_1)$ and $X(t_2)$ are linearly independent

Autocorrelation $\rho(t_1, t_2)$ is the correlation between random variables $X(t_1)$ and $X(t_2)$, $t_1 \in \mathscr{T}$, $t_2 \in \mathscr{T}$

- $\rho(t_1, t_2) = \rho(t_2, t_1)$
- $\rho(t, t) = 1$
- $|\rho(t_1, t_2)| \leq 1$
- $X(t_1)$ and $X(t_2)$ are independent $\Rightarrow \rho(t_1, t_2) = 0$
- $\rho(t_1, t_2) = 0 \Rightarrow X(t_1)$ and $X(t_2)$ are linearly independent

Cross-covariance and Cross-correlation

Cross-covariance of 2-D time series (X, Y) is a deterministic function $c_{XY}(t_1, t_2)$, $t_1 \in \mathscr{T}$, $t_2 \in \mathscr{T}$, defined as

$$c_{XY}(t_1, t_2) = cov(X(t_1), Y(t_2))$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x(t_1) - m_X(t_1))(y(t_2) - m_Y(t_2)) f_{XY}(x, y; t_1, t_2) dxdy$$

Cross-correlation of 2-D time series (X, Y) is the cross-covariance of standardized time series X and Y:

$$\rho_{XY}(t_1, t_2) = cov\left[\frac{X(t_1) - m_X(t_1)}{\sigma_X(t_1)}, \frac{Y(t_2) - m_Y(t_2)}{\sigma_Y(t_2)}\right] = \frac{c_{XY}(t_1, t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}$$

$\rho_{XY}(t_1, t_2)$ is not symmetric; $|\rho_{XY}(t_1, t_2)| \le 1$ for all $t_1, t_2 \in \mathscr{T}$;
$\rho_{XY}(t, t)$ can be less than 1, $t \in \mathscr{T}$

Multivariate CDF and PDF

Choose some finite set of times $t_1, ..., t_k \in \mathscr{T}$

For a set of corresponding random variables $X(t_1), ..., X(t_k)$ we can specify their k-th order joint probability distribution functions

The k-th order CDF of a random signal X is defined as

$$\begin{aligned} F(x_1, ..., x_k; t_1, ..., t_k) &= F_{X(t_1)...X(t_k)}(x_1, ..., x_k) \\ &= P(X(t_1) < x_1 \& ... \& X(t_k) < x_k) \end{aligned}$$

The k-th order PDF of a random signal X is defined as the k-th derivative of the k-th order CDF:

$$f(x_1, ..., x_k; t_1, ..., t_k) = \frac{\partial^k F(x_1, ..., x_k; t_1, ..., t_k)}{\partial x_1 ... \partial x_k}$$

Strict-Sense Stationary Process

A random process X is called stationary if its statistical properties do not change by time

We need to specify which statistical properties are to be time-invariant

Definition

A random process X is called strict-sense (or strong) stationary (SSS) if all its finite order distributions are time invariant, i.e. joint probability distributions of

$$X(t_1), ..., X(t_k) \quad \text{and} \quad X(t_1 + \tau), ..., X(t_k + \tau)$$

are the same for all k, all $t_1, ..., t_k \in \mathscr{T}$ and all time shifts $\tau \in \mathscr{T}$:

$$f(x_1, ..., x_k; t_1, ..., t_k) = f(x_1, ..., x_k; t_1 + \tau, ..., t_k + \tau)$$

k-th Order Stationarity

Definition

A random process X is called k-th order stationary if its distributions up to k-th order are time-shift invariant:

$$f(x_1, ..., x_j; t_1, ..., t_j) = f(x_1, ..., x_j; t_1 + \tau, ..., t_j + \tau)$$

for all $j = 1, ..., k$, all $t_1, ..., t_j \in \mathscr{T}$ and all time shifts $\tau \in \mathscr{T}$

1-st order stationarity:

$$f(x, t) = f(x, t + \tau), \quad \forall \tau, t \in \mathscr{T}, \forall x \in \mathscr{X}$$

2-nd order stationarity:

$$f(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1+\tau, t_2+\tau), \quad \forall \tau, t_1, t_2 \in \mathscr{T}, \forall x_1, x_2 \in \mathscr{X}$$

Properties of 2-nd Order Stationary Processes

- First-order distributions $f(x,t)$ are the same for all $t \in \mathscr{T}$:

$$f(x,t) = f(x,t+\tau) = f(x), \quad \forall \tau \in \mathscr{T}$$

Particularly,

$$m(t) = m = const$$

$$\sigma^2(t) = \sigma^2 = const$$

2-nd order stationarity $\Rightarrow$ 1-st order stationarity

- Second-order distribution $f(x_1, x_2; t_1, t_2)$ depends only on time shift $t_2 - t_1$:

$$f(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1 + \tau, t_2 + \tau), \quad \forall \tau \in \mathscr{T}$$

$$\tau := -t_1 \Rightarrow f(x_1, x_2; t_1, t_2) = f(x_1, x_2; 0, t_2 - t_1) = f(x_1, x_2; \tau)$$

Properties of 2-nd Order Stationary Processes

- Autocovariance $c(t_1, t_2)$ and autocorrelation $\rho(t_1, t_2)$ depend only on time shift $\tau = t_2 - t_1$:

$$c(t_1, t_2) = cov(X(t_1), X(t_2)) = cov(X(0), X(t_2 - t_1))$$
$$= c(0, t_2 - t_1) = c(\tau)$$

$$\rho(t_1, t_2) = \frac{c(t_1, t_2)}{\sigma(t_1)\sigma(t_2)} = \frac{c(0, t_2 - t_1)}{\sigma(0)\sigma(t_2 - t_1)} = \rho(0, t_2 - t_1) = \rho(\tau)$$

Properties of autocovariance and autocorrelation functions (ACF) of 2-nd order stationary process:

- $c(\tau) = c(-\tau), \qquad \rho(\tau) = \rho(-\tau)$
- $|c(\tau)| \leq c(0) = \sigma^2, \qquad |\rho(\tau)| \leq \rho(0) = 1$
- $\rho(\tau) = \dfrac{c(\tau)}{\sigma^2} = \dfrac{c(\tau)}{c(0)}$

Wide-Sense Stationary Random Process

Definition

A random process X is called wide-sense (or weak) stationary (WSS) if its mean and autocovariance function are time-shift invariant:

$$1)\ \mathrm{M}[X(t)] = \mathrm{M}[X(t+\tau)], \quad \forall \tau \in \mathscr{T}$$

2) $c(t_1, t_2)$ is a function only of time difference $\tau = t_2 - t_1$

- Expectation and variance of WSS process are constants:

$$m(t) = m = const$$

$$\sigma^2(t) = c(t,t) = c(0) = const$$

- Autocorrelation function $r(t_1, t_2)$ of WSS process depends only on time shift $\tau = t_2 - t_1$:

$$\rho(t_1, t_2) = \rho(0, t_2 - t_1) = \rho(\tau)$$

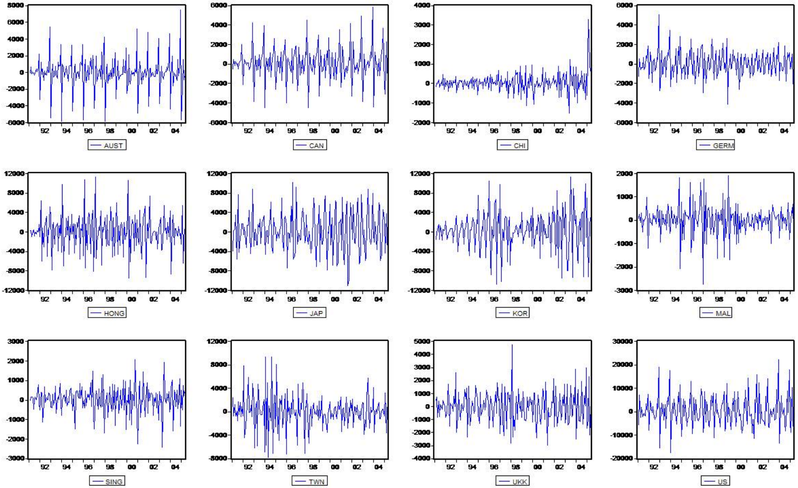
Notes on WSS Processes

- Wide-sense stationarity is a particular case of 2-nd order stationarity
- Autocovariance and autocorrelation functions of WSS process are functions of $|t_2 - t_1|$:

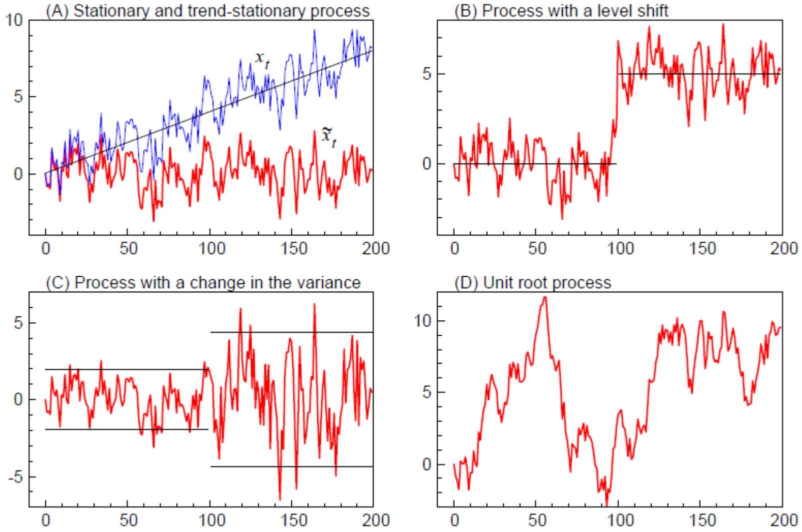
$$c(\tau) = c(-\tau), \quad \rho(\tau) = \rho(-\tau)$$

- SSS $\Rightarrow$ 2-nd order stationarity $\Rightarrow$ WSS. The converse is not necessarily true
- WSS process is not necessarily 1-st order stationary
- 1-st order stationary process is not necessarily WSS process
- SSS process sometimes is called "stationary process"

Stationary Processes. Illustrations

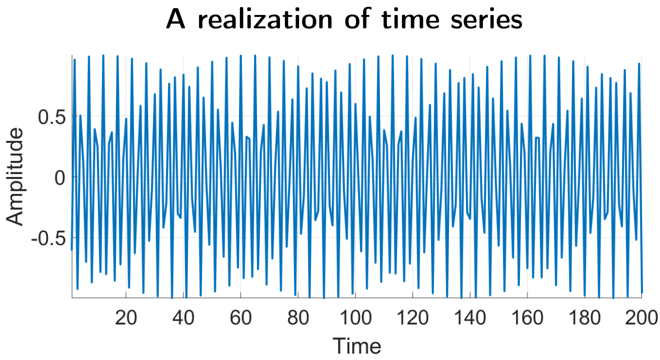


Non-Stationary Processes. Illustrations



Example 1

Consider a time series $X(t) = \sin(Vt)$, where V is a random variable, $V \sim R(0, 2\pi)$ and $t = 1, 2, ...$



For each realization $\{x(t), t = 1, 2, ...\}$ the frequency v is constant over time

Example 1. Check WSS

Expectation of time series X:

$$m(t) = \mathrm{M}[X(t)] = \mathrm{M}[\sin(Vt)] = \frac{1}{2\pi}\int\limits_0^{2\pi} \sin(vt)dv = \frac{-1}{2\pi t}\cos(vt)\Big|_0^{2\pi} = 0$$

Variance of time series X:

$$\sigma^2(t) = \mathrm{D}[X(t)] = \mathrm{D}[\sin(Vt)] = \frac{1}{2\pi}\int\limits_0^{2\pi} \sin^2(vt)dv$$

$$= \frac{1}{4\pi}\int\limits_0^{2\pi}(1-\cos(2vt))dv = \frac{1}{4\pi}\left(v - \frac{\sin(2vt)}{2t}\right)\Big|_0^{2\pi}$$

$$= \frac{1}{4\pi}(2\pi - 0) = \frac{1}{2}$$

Example 1. Check WSS

Autocovariance of time series X:

$$c(t, t+\tau) = \mathrm{M}[X(t)X(t+\tau)] = \frac{1}{2\pi}\int\limits_0^{2\pi} \sin(vt)\sin(vt+v\tau)dv$$

$$= \frac{1}{4\pi}\int\limits_0^{2\pi} (\cos(v\tau) - \cos(2vt + v\tau))dv$$

$$= \frac{1}{4\pi}\left(\frac{\sin(v\tau)}{\tau} - \frac{\sin(2vt+v\tau)}{2t+\tau}\right)\Bigg|_{v=0}^{v=2\pi} = 0, \quad \tau \neq 0$$

Note, that $c(t, t+\tau) = 0$ (and the time series is WSS) if $t = 1, 2, \ldots$

But if $t \in [0, +\infty)$, then $c(t, t+\tau)$ depends on τ and t and the process X is not WSS

Example 1. Check SSS

Firstly, check 1-st order stationarity

Let's take two time points $t_1 \in \{1, 2, ...\}$ and $t_2 \in \{1, 2, ...\}$ and compare distributions of random variables

$$X(t_1) = \sin(Vt_1) \quad \text{and} \quad X(t_2) = \sin(Vt_2)$$

It can be shown that if random variable $V \sim R(0, 2\pi)$, then random variables $\sin(V), \sin(2V), ...$ have arcsine distribution

So, the random variables $X(1), X(2), ...$ have the same distribution $\Rightarrow$ the time series X is 1-st order stationary

Example 1. Check SSS

Check 2-nd order stationarity

The bivariate CDF at time points $t_1 = 1$ and $t_2 = 2$:

$$\begin{aligned} F(x_1, x_2; 1, 2) &= P(X(1) < x_1 \& X(2) < x_2) \\ &= P(\sin V < x_1 \& \sin 2V < x_2) \end{aligned}$$

The bivariate CDF at time points $t_1 = 2$ and $t_2 = 3$:

$$\begin{aligned} F(x_1, x_2; 2, 3) &= P(X(2) < x_1 \& X(3) < x_2) \\ &= P(\sin 2V < x_1 \& \sin 3V < x_2) \end{aligned}$$

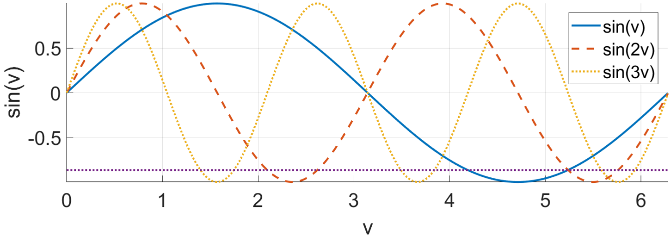
For 2-nd order stationarity process these CDFs must be equal

Example 1. Check SSS

Assume $x_1 = x_2 = -\frac{\sqrt{3}}{2}$:

$$F(x_1, x_2; 1, 2) = P\left(\sin V < -\frac{\sqrt{3}}{2} \& \sin 2V < -\frac{\sqrt{3}}{2}\right) = 0$$

$$F(x_1, x_2; 2, 3) = P\left(\sin 2V < -\frac{\sqrt{3}}{2} \& \sin 3V < -\frac{\sqrt{3}}{2}\right) > 0$$



Hence, X is not 2-nd order stationary time series $\Rightarrow$ X is not SSS

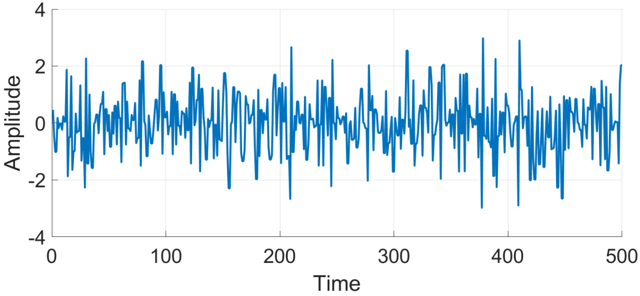
Example 2

Consider a time series

$$X(t) = X(t-1)\cos\frac{\pi t}{2} + U(t)\sin\frac{\pi t}{2}, \quad X(1) = U(1)$$

where $U(1), U(2), \ldots$ is are i.i.d. random variables, $U(t) \sim N(0,1)$ for all $t = 1, 2, \ldots$

A realization of time series



Example 2. Check 1-st Order Stationarity

Expectation of time series X:

$$m(1) = \mathrm{M}[X(1)] = \mathrm{M}[U(1)] = 0$$

$$m(t) = \mathrm{M}\left[X(t-1)\cos\frac{\pi t}{2} + U(t)\sin\frac{\pi t}{2}\right]$$

$$= \mathrm{M}[X(t-1)]\cos\frac{\pi t}{2} + \mathrm{M}[U(t)]\sin\frac{\pi t}{2} = 0 \quad \text{by induction}$$

Variance of time series X:

$$\sigma^2(1) = \mathrm{D}[X(1)] = \mathrm{D}[U(1)] = 1$$

$$\sigma^2(t) = \mathrm{D}\left[X(t-1)\cos\frac{\pi t}{2} + U(t)\sin\frac{\pi t}{2}\right]$$

$$= \mathrm{D}[X(t-1)]\cos^2\frac{\pi t}{2} + \mathrm{D}[U(t)]\sin^2\frac{\pi t}{2} = 1 \quad \text{by induction}$$

All random variables $X(1), X(2), \ldots$ are distributed as $N(0,1)$

$\Rightarrow$ the time series X is 1-st order stationary

Example 2. Check WSS

Autocovariance of time series X at times t and $t+1$:

$$
\begin{aligned}
c(t,t+1) &= \mathrm{M}[X(t)X(t+1)] = \\
&= \mathrm{M}\left[X(t)\left(X(t)\cos\frac{\pi(t+1)}{2} + U(t+1)\sin\frac{\pi(t+1)}{2}\right)\right] \\
&= \mathrm{M}\left[-X^2(t)\sin\frac{\pi t}{2} + X(t)U(t+1)\cos\frac{\pi t}{2}\right] \\
&= -\sin\frac{\pi t}{2}\mathrm{M}[X^2(t)] + \cos\frac{\pi t}{2}\mathrm{M}[X(t)U(t+1)] \\
&= -\sin\frac{\pi t}{2}(\mathrm{D}[X(t)] + \mathrm{M}[X(t)]^2) + \cos\frac{\pi t}{2}\mathrm{M}[X(t)]\mathrm{M}[U(t+1)] \\
&= -\sin\frac{\pi t}{2}
\end{aligned}
$$

Autocovarinace $c(t,t+1) \in \{-1,0,1\}$ and depends on t

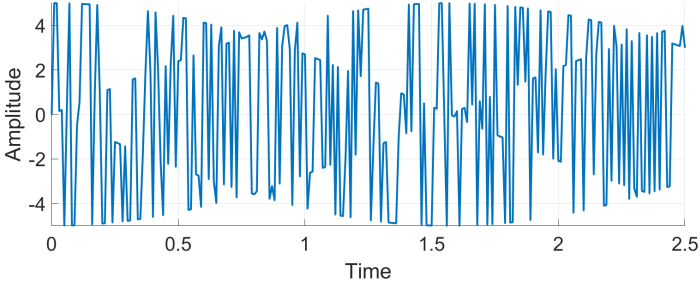
$\Rightarrow$ time series X is not WSS

Example 3

Consider a random process $\{X(t), t \in [0, \infty)\}$, which has one of four equally probable outcomes at any time $t \in [0, \infty)$:

$$X(t) \in \{-5\sin t, 5\sin t, -5\cos t, 5\cos t\}$$

A realization of random process



Example 3. Check WSS

Expectation of process X:

$$m(t) = \mathrm{M}[X(t)] = \frac{1}{4}(-5\sin t + 5\sin t - 5\cos t + 5\cos t) = 0$$

Variance of process X:

$$\sigma^2(t) = \mathrm{D}[X(t)] = \frac{25}{4}(\sin^2 t + \sin^2 t + \cos^2 t + \cos^2 t) = \frac{25}{2}$$

Autocovariance of process X:

$$X(t+\tau) \in \{-5\sin(t+\tau), 5\sin(t+\tau), -5\cos(t+\tau), 5\cos(t+\tau)\}$$

$$c(t, t+\tau) = \mathrm{M}[X(t)X(t+\tau)]$$
$$= \frac{1}{16}(25\sin(t)\sin(t+\tau) + \ldots + 25\cos(t)\cos(t+\tau)) = 0,\ \ \tau \neq 0$$

Autocovarinace $c(t, t+\tau)$ doesn't depend on t

$\Rightarrow$ process X is WSS

Example 3. Check SSS

$$X(t) \in \{-5\sin t, 5\sin t, -5\cos t, 5\cos t\}$$

Firstly, check 1-st order stationarity

Let's take two time points $t_1 = 0$ and $t_2 = \frac{\pi}{4}$ and compare distributions of random variables $X(t_1)$ and $X(t_2)$:

$$X(t_1) \in \{0, -5, 5\}$$

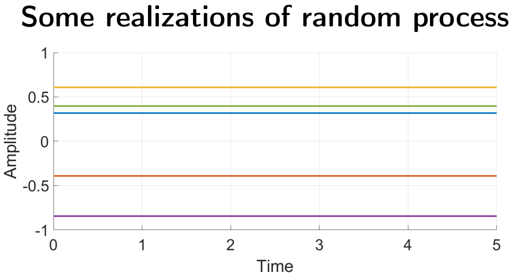
$$X(t_2) \in \left\{-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right\}$$

These distributions are different (even though their mean is the same), hence, the process X is not 1-st order stationary

$\Rightarrow$ X is not SSS process

Example 4

Consider a random process $X(t) = A$, where $A \sim R(-1, 1)$



Expectation: $m(t) = \mathrm{M}[X(t)] = \mathrm{M}[A] = 0$
Autocovariance: $c(t, t+\tau) = \mathrm{M}[X(t)X(t+\tau)] = \mathrm{M}[A^2] = 1/3$
1-st order PDFs: $f(x, t) = f(x, t+\tau) = R(-1, 1)$
2-nd order PDFs: $f(x_1, x_2; t_1, t_2) = R(-1, 1) \times R(-1, 1), \ \forall t_1, t_2$

Inductively, process X is SSS

Jointly Stationary Processes

Two random processes X and Y are jointly WSS (or simply jointly stationary), if

1. X is WSS
2. Y is WSS
3. $c_{XY}(t_1, t_2)$ depends only on $\tau = t_2 - t_1$

Time shift τ is called time lag

Properties of cross-covariance function $c_{XY}(\tau)$ and cross-correlation function (XCF) $\rho_{XY}(\tau)$ of jointly stationary processes X and Y:

- $c_{XY}(\tau)$ and $\rho_{XY}(\tau)$ are not symmetric with respect to zero
- $|\rho_{XY}(\tau)| \leq 1$ for all $\tau \in \mathscr{T}$
- $\rho_{XY}(0)$ can be less than 1

Averaging of Stationary Process

The expectation m of stationary (WSS) process X doesn't depend on t:

$$m = m(t) = \mathrm{M}[X(t)] = \int\limits_{-\infty}^{+\infty} x f(x,t) dx \quad \text{for any time } t \in \mathscr{T}$$

In practice, the PDF $f(x,t)$ is usually unknown

How to estimate the expectation m using the observations of stationary process X?

Stationary process can be averaged in two ways:

- Ensemble averaging
- Time averaging

Ensemble Averaging and Time Averaging

Ensemble average of a random process X over realizations $x_1(t), ..., x_n(t)$, $t \in \mathscr{T}$:

$$\hat{m}_N(t) = \frac{1}{N} \sum_{n=1}^{N} x_n(t)$$

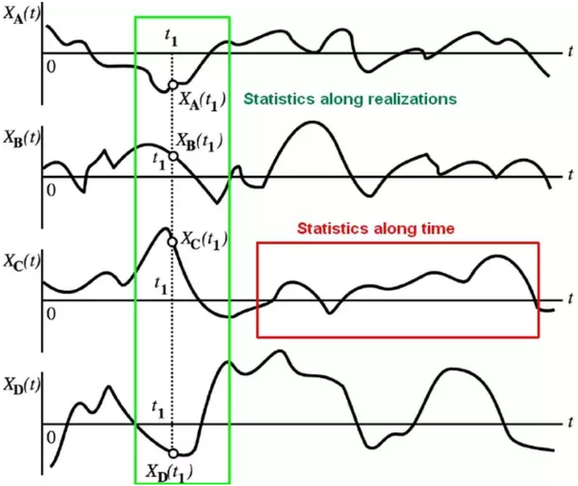
Time average of continuous-time random process X over an interval (t_1, t_2):

$$\bar{X} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} X(t) dt$$

Time average of time series X over a set of time moments $\{t_1, ..., t_2\}$:

$$\bar{X} = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} X(t)$$

Averaging of Stationary Process. Illustration



Time Average as Estimator

Ensemble average $\hat{m}_N(t)$ at any time moment $t \in \mathscr{T}$ is a consistent estimator of m:

$$\lim_{N\to\infty} \hat{m}_N(t) = m$$

In practice, we have only one realization $\{x(t), t \in \mathscr{T}\}$, of stationary process X

In this case, the only way to estimate the expectation m is to use time averaging of the realization $\{x(t), t \in \mathscr{T}\}$

Can the expectation m be estimated using time averaging of a single realization $\{x(t), t \in \mathscr{T}\}$?

To answer we need to analyse statistical properties of random variable $\bar{X}$ as the estimator of expectation m: unbiasedness and consistency

Unbiasedness of Discrete-Time Average

Consider WSS time series $\{X(t),\, t = 1, 2, ...\}$

Time averaged mean: $\bar{X}_T = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} X(t)$

Expectation of $\bar{X}_T$:

$$\mathrm{M}[\bar{X}_T] = \mathrm{M}\left[\frac{1}{T}\sum_{t=1}^{T} X(t)\right] = \frac{1}{T}\sum_{t=1}^{T}\mathrm{M}[X(t)] = \frac{1}{T}Tm = m$$

Hence, the time averaged mean $\bar{X}_T$ is unbiased estimator of m

Sufficient condition of consistency:

$$\lim_{T\to\infty} \mathrm{M}[\bar{X}_T] = m$$

$$\lim_{T\to\infty} \mathrm{D}[\bar{X}_T] = 0$$

Variance of Discrete-Time Average

The variance of $\bar{X}_T$:

$$\mathrm{D}[\bar{X}_T] = \mathrm{M}[(\bar{X}_T - m)^2] = \mathrm{M}\left[\left(\frac{1}{T}\sum_{t=1}^{T} X(t) - m\right)^2\right]$$

$$= \mathrm{M}\left[\left(\frac{1}{T}\sum_{t=1}^{T}(X(t) - m)\right)^2\right] = \frac{1}{T^2}\mathrm{M}\left[\left(\sum_{t=1}^{T}(X(t) - m)\right)^2\right]$$

$$= \frac{1}{T^2}\mathrm{M}\left[\sum_{t=1}^{T}\sum_{s=1}^{T}(X(t) - m)(X(s) - m)\right]$$

$$= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathrm{M}[(X(t) - m)(X(s) - m)] = \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} c(t,s)$$

Consistency of Discrete-Time Average

X is WSS $\Rightarrow$ the ACF $c(t,s)$ depends only on time shift $\tau = t - s$.

So, we'll sum $\sum\limits_{s=1}^{T} c(t,s)$ over diagonals of covariance matrix $T \times T$:

$$\mathrm{D}[\bar{X}_T] = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} c(t,s) = \frac{1}{T^2} \left[Tc(0) + \sum_{\tau=1}^{T-1} 2(T-\tau)c(\tau) \right]$$

$$= \frac{c(0)}{T} + \frac{2}{T^2} \sum_{\tau=1}^{T-1} c(\tau)(T-\tau)$$

The time averaged mean $\bar{X}_T$ is consistent, if

$$\lim_{T\to\infty} \mathrm{D}[\bar{X}_T] = \lim_{T\to\infty} \frac{2}{T^2} \sum_{\tau=1}^{T-1} c(\tau)(T-\tau) = 0$$

The consistency of $\bar{X}_T$ is related to the properties of ACF $c(\tau)$

Time Averaging for Continuous Processes

Consider continuous-time WSS process $X(t)$, $t \in (-\infty, \infty)$

The time averaged mean: $\bar{X}_T = \dfrac{1}{T} \displaystyle\int\limits_{-T/2}^{T/2} X(t)dt$

The expectation of $\bar{X}_T$**:**

$$\mathrm{M}[\bar{X}_T] = \mathrm{M}\left[\frac{1}{T}\int\limits_{-T/2}^{T/2} X(t)dt\right] = \frac{1}{T}\int\limits_{-T/2}^{T/2} \mathrm{M}[X(t)]dt = \frac{1}{T}Tm = m$$

Hence, the time averaged mean $\bar{X}_T$ is unbiased estimator of m

Variance of Continuous-Time Average

The variance of $\bar{X}_T$:

$$\mathrm{D}[\bar{X}_T] = \mathrm{M}[(\bar{X}_T - m)^2] = \mathrm{M}\left[\left(\frac{1}{T}\int_{-T/2}^{T/2} X(t)dt - m\right)^2\right]$$

$$= \frac{1}{T^2}\mathrm{M}\left[\left(\int_{-T/2}^{T/2} (X(t) - m)dt\right)^2\right]$$

$$= \frac{1}{T^2}\mathrm{M}\left[\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} (X(t) - m)(X(s) - m)ds\right]$$

$$= \frac{1}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} c(t,s)dtds = \frac{1}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} c(t-s)dtds$$

We can simplify double integration using the substitution $\tau = t - s$

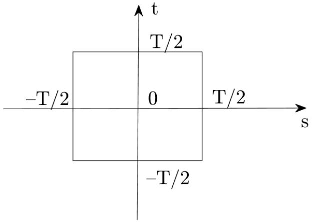
Variance of Continuous-Time Average

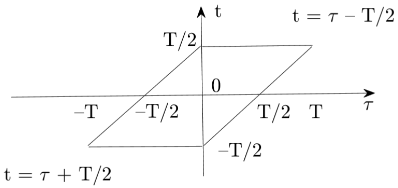
Boundaries:

$$s \in \left[-\frac{T}{2}, \frac{T}{2}\right], \ t \in \left[-\frac{T}{2}, \frac{T}{2}\right] \ \Rightarrow \ \tau \in [-T, T]$$

For a fixed $\tau \in [-T, T]$:

$$t = \tau + s \in \left[\max\left(-\frac{T}{2}, \tau - \frac{T}{2}\right), \min\left(\frac{T}{2}, \tau + \frac{T}{2}\right)\right]$$





Variance of Continuous-Time Average

$$
\begin{aligned}
\mathrm{D}[\bar{X}_T] &= \frac{1}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} c(t-s)\,dt\,ds \qquad\qquad [t-s=\tau] \\
&= \frac{1}{T^2}\left(\int_{-T}^{0} d\tau \int_{-T/2}^{\tau+T/2} c(\tau)dt + \int_{0}^{T} d\tau \int_{\tau-T/2}^{T/2} c(\tau)dt\right) \\
&= \frac{1}{T^2}\left(\int_{-T}^{0} c(\tau)d\tau \int_{-T/2}^{\tau+T/2} dt + \int_{0}^{T} c(\tau)d\tau \int_{\tau-T/2}^{T/2} dt\right) \\
&= \frac{1}{T^2}\left(\int_{-T}^{0} c(\tau)(T+\tau)d\tau + \int_{0}^{T} c(\tau)(T-\tau)d\tau\right) \\
&= \frac{1}{T^2}\int_{-T}^{T} c(\tau)(T-|\tau|)d\tau
\end{aligned}
$$

The consistency of $\bar{X}_T$ is related to the properties of ACF $c(\tau)$

Ergodic Processes

A stationary process X is called ergodic if its statistical properties can be deduced from a single, sufficiently long realization of the process

We need to specify which statistical properties can be deduced

Definition

A stationary process X is called mean-ergodic if its time average converges to the ensemble average:

$$\bar{X}_T \to m$$

in the squared mean sense, i.e.

$$\lim_{T\to\infty} \mathrm{M}[(\bar{X}_T - m)^2] = 0$$

Consistency of Time Average and Mean-Ergodicity

As soon as time averaged mean $\bar{X}_T$ is unbiased estimator of m, the mean-ergodicity condition

$$\lim_{T\to\infty} \mathrm{M}[(\bar{X}_T - m)^2] = 0$$

is equivalent to the sufficient condition of consistency of $\bar{X}_T$:

$$\lim_{T\to\infty} \mathrm{D}[\bar{X}_T] = 0$$

Stationary process X is mean-ergodic $\Rightarrow$ $\bar{X}_T$ is a consistent estimator of m

The mean-ergodicity of stationary process X is related to the properties of its ACF $c(\tau)$

Ergodic Theorem

Ergodic theorem

Let's $\{x(t), t = 1, 2, ...\}$ is a some realization of a WSS time series X. Then the time average $\bar{x}_T$ converges to its expectation m

$$\lim_{T\to\infty} \bar{x}_T = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} x(t) = m$$

iff the time average of its autocovariation function converges to 0 when $T \to \infty$:

$$\lim_{T\to\infty} \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} c(t, s) = 0$$

Ergodic theorem gives necessary and sufficient condition of mean ergodicity

Sufficient Conditions of Mean Ergodicity

Criterion of mean-ergodicity for discrete-time WSS process:

$$\lim_{T\to\infty} \mathrm{D}[\bar{X}_T] = \lim_{T\to\infty} \frac{2}{T^2} \sum_{\tau=1}^{T-1} c(\tau)(T-\tau) = 0$$

Some sufficient conditions of ergodicity:

- Independence of $X(t)$ and $X(t+\tau)$ for all t when increasing the time shift τ
- $c(\tau) = 0$ for all $\tau > \tau_0$
- $\lim\limits_{\tau\to\infty} c(\tau) = 0$
- $\lim\limits_{T\to\infty} \sum_{\tau=1}^{T} |c(\tau)| = const < \infty$

$$\frac{2}{T^2} \sum_{\tau=1}^{T-1} c(\tau)(T-\tau) = \frac{2}{T} \sum_{\tau=1}^{T-1} c(\tau) \frac{T-\tau}{T} \le \frac{2}{T} \sum_{\tau=1}^{T-1} |c(\tau)|$$

Slutsky's Theorem (1938)

Slutsky's Theorem

A continuous-time WSS process $\{X(t), t \in (-\infty, \infty)\}$ is mean-ergodic iff

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} c(\tau) d\tau = 0$$

A discrete-time WSS process $\{X(t), t = 1, 2, ...\}$ is mean-ergodic iff

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\tau=1}^{T} c(\tau) = 0$$

Slutsky's theorem gives necessary and sufficient condition of mean-ergodicity

Estimation of Autocovariance

Covariance of WSS time series X:

$$c(\tau) = \mathrm{M}[(X(t) - m)(X(t+\tau) - m)], \quad \forall t \in \{1, 2, ...\}$$

The covariance of process X at time shift τ is equal to the mean of process $Y_\tau(t) = (X(t) - m)(X(t+\tau) - m)$, $t \in \{1, 2, ...\}$:

$$c(\tau) = \mathrm{M}[Y_\tau(t)], \quad \tau = 0, 1, 2...$$

If m is unknown, its estimation is used

Ensemble averaged covariance:

$$\hat{c}_N(\tau) = \frac{1}{N} \sum_{n=1}^{N} (x_n(t) - \hat{m}_N)(x_n(t+\tau) - \hat{m}_N)$$

Time averaged covariance over set $\{1, ..., T\}$:

$$\tilde{c}_T(\tau) = \frac{1}{T-\tau} \sum_{t=1}^{T-\tau} (X(t) - \bar{X}_T)(X(t+\tau) - \bar{X}_T)$$

Autocovariance-Ergodic Processes

A WSS random process is autocovariance-ergodic (ergodic with respect to autocovariance) if its autocovariance function can be obtained from its any single realization

Definition

A WSS random process X is called autocovariance-ergodic if its time average estimate of autocovariance converges to the true autocovariance for all $\tau \in \mathscr{T}$:

$$\tilde{c}_T(\tau) \to c(\tau)$$

in the squared mean sense, i.e.

$$\lim_{T\to\infty} \mathrm{M}\left[(\tilde{c}_T(\tau) - c(\tau))^2\right] = 0 \quad \forall \tau \in \mathscr{T}$$

Criterion of Covariance-ergodicity

Criterion of mean-ergodicity of WSS time series X:

$$\lim_{T\to\infty} \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} c_X(t,s) = 0$$

Covariance-ergodicity of time series X is equivalent to mean-ergodicity of all time series $Y_0, Y_1, \ldots$, where $Y_\tau = \{Y_\tau(t), t \in \{1,2,\ldots\}\}$ and

$$Y_\tau(t) = (X(t) - m)(X(t+\tau) - m),\ \ \tau \in \{0,1,\ldots\}$$

Criterion of covariance-ergodicity of WSS time series X:

$$\lim_{T\to\infty} \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} c_{Y_\tau}(t,s) = 0, \quad \forall \tau = 0,1,\ldots$$

Notes on Covariance-ergodicity

- In order to check whether $\{Y_\tau(t), t = 1, 2, ...\}$, is mean-ergodic, it is necessary to require it be WSS for all $\tau = 0, 1, ...$
- Autocovariance of process $Y_\tau(t)$ is the expectation that depends on random variables $X(t)$ in four points in time. Therefore, it is not enough to require that X be WSS, but a stronger requirement of 4-th order stationary is needed
- Variance-ergodicity of process X is a particular case of covariance-ergodicity when $\tau = 0$
- Cross-covariance-ergodicity of processes X and Y means that their cross-covariance function can be obtained from any single realization of two-dimensional process (X, Y)

Why Ergodicity?

Random process is a set of random variables $X(t_1), X(t_2), \ldots$

We need ensemble of realizations $x_1(t), \ldots, x_N(t)$, $t \in \mathscr{T}$, to estimate statistical characteristics of these random variables

What is wrong with averaging over the ensemble of realizations?

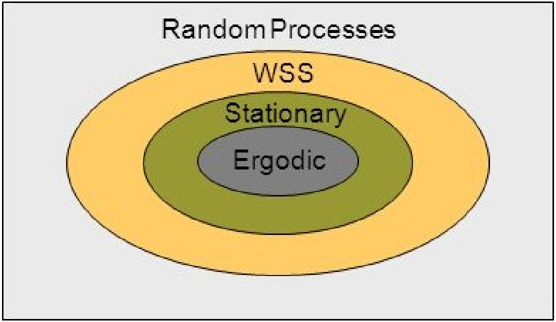
- A huge amount of data is required
- Sometimes it is not just laborious, but impossible to obtain many realizations of the random process. Many experiments cannot be repeated even twice (e.g. change of climate on Earth, stock time series)

Ergodicity allow us to obtain statistical characteristics of random process X from any of its single realizations $\{x(t), t \in \mathscr{T}\}$

For ergodic processes the averaging over time can be used

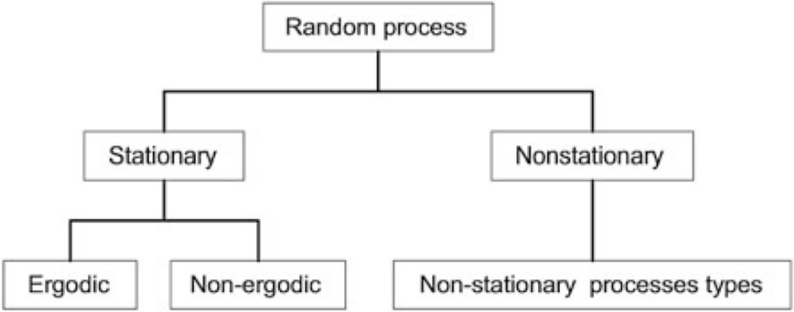
Stationarity and Ergodicity

Ergodicity always mean stationarity. There is no ergodicity without stationarity



Random process can be stationary and not ergodic

Types of Processes



Types of non-stationarities:

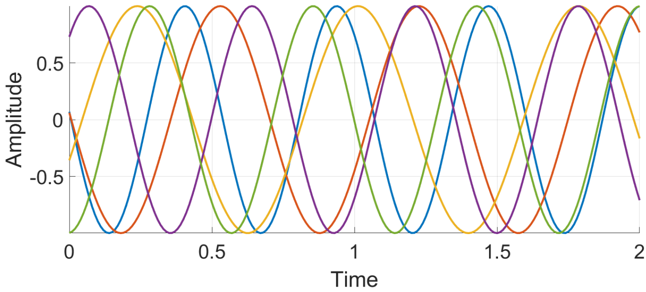
- Non-stationarity in mean (trend)
- Non-stationarity in variance
- Non-stationarity in autocovariance
- ...

Mean and Covariance Ergodicity. Example

Consider a random process $X(t) = \cos(\omega t + \varphi)$, $t \in [0, \infty)$, where ω and φ are random variables:

$$\omega \sim R(8; 12), \qquad \varphi \sim R(0; 2\pi)$$

Some realizations of random process



Each realization has different frequency ω (but constant over time) and different initial phase φ

Example. Check WSS

Conditions of WSS:

- Expectation $m = const$

$$m = \mathrm{M}[\cos(\omega t + \varphi)] = \int_0^{2\pi} \int_8^{12} \cos(\omega t + \varphi) \frac{1}{2\pi} \frac{1}{4} d\omega d\varphi = 0$$

- Variance $\sigma^2 = const$

$$\sigma^2 = \mathrm{D}[\cos(\omega t + \varphi)] = \mathrm{M}[\cos^2(\omega t + \varphi)] = 1/2$$

- Covariance $c(t, t+\tau)$ depends only on τ

$$c(t, t+\tau) = \mathrm{M}[X(t)X(t+\tau)] = \mathrm{M}[\cos(\omega t + \varphi)\cos(\omega t + \omega\tau + \varphi)]$$
$$= \frac{1}{8\tau}\left(\sin(12\tau) - \sin(8\tau)\right), \quad \tau \neq 0$$

Expectation $m = const$, ACF $c(t, t+\tau)$ depends only on τ

$\Rightarrow$ the process X is WSS

Example. Check Mean-Ergodicity

Let's $x(t)$ to be a realization of process X

Check if the process X is ergodic w.r.t. mean:

$$\bar{x}_T = \frac{1}{T}\int\limits_{-T/2}^{T/2} \cos(\omega t + \varphi)dt = \frac{1}{T}\frac{1}{\omega}\sin(\omega t + \varphi)\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{\omega T}\left(\sin\left(\frac{\omega T}{2} + \varphi\right) - \sin\left(-\frac{\omega T}{2} + \varphi\right)\right)$$

$$= \frac{2}{\omega T}\sin\left(\frac{\frac{\omega T}{2} + \varphi + \frac{\omega T}{2} - \varphi}{2}\right)\cos\left(\frac{\frac{\omega T}{2} + \varphi - \frac{\omega T}{2} + \varphi}{2}\right)$$

$$= \frac{2}{\omega T}\sin\left(\frac{\omega T}{2}\right)\cos\varphi \to 0 \quad \text{when } T \to \infty$$

$\bar{x}_T \to 0 = m \Rightarrow$ the process X is ergodic w.r.t. mean

Example. Check Autocovariance-Ergodicity

Check if the process is ergodic w.r.t. covariance:

$$\begin{aligned}
\tilde{c}_T(t,t+\tau) &= \frac{1}{T}\int_{-T/2}^{T/2} \cos(\omega t+\varphi)\cos(\omega(t+\tau)+\varphi)dt \\
&= \frac{1}{2T}\int_{-T/2}^{T/2} (\cos(2\omega t+\omega\tau+2\varphi)+\cos(\omega\tau))dt \\
&= \frac{1}{2T}\left[\frac{1}{2\omega}\sin(2\omega t+\omega\tau+2\varphi)+t\cos(\omega\tau)\right]_{-T/2}^{T/2} \\
&= \frac{1}{4\omega T}\left[\sin(\omega T+\omega\tau+2\varphi)-\sin(-\omega T+\omega\tau+2\varphi)\right]+\frac{1}{2}\cos(\omega\tau) \\
&\to \frac{1}{2}\cos(\omega\tau) \quad \text{when } T\to\infty
\end{aligned}$$

$\tilde{c}_T(\tau) \to \frac{1}{2}\cos(\omega\tau) \neq c(\tau) = \frac{1}{8\tau}\left(\sin(12\tau)-\sin(8\tau)\right)$

$\Rightarrow$ the process X is not ergodic w.r.t. covariance

Why Some WSS Processes Are Not Ergodic?

Ergodicity is the property by which each realization of a given stationary process is a complete and independent representative of all possible realizations of the process

For non-ergodic process its realizations carry information only about the given realization and not about the underlying process

For non-ergodic processes the amplitudes at further time moments don't carry any new information about process (because of high correlation with amplitudes at previous time moments)

For ergodic processes a strength of dependence between random variables in the process diminishes the farther apart they become. It results to retrieving new information from observations at further time moments

Notes on Stationarity and Ergodicity Tests

When determining the stationarity and ergodicity we used the known model of the process X

How to test stationarity if the model of the process is unknown?

- Visual analysis of time series
  Look at any obvious trends, seasonality, changes in variance, disruptions etc.
- Moving summary statistics
  Partitioning the time series into time intervals and check for obvious or significant differences in summary statistics
- Statistical tests
  Perform statistical tests under the null hypothesis of stationarity

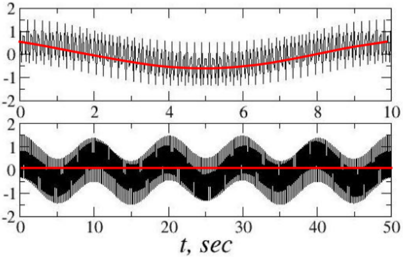
Notes on Stationarity and Ergodicity Tests

All definitions of stationarity and ergodicity are applied to the processes that can be observed for an infinitely long time

However, in practice, any process can be observed only during a finite time interval. It might lead one to wrong conclusions about statistical properties of signal

The mean of process seems to be floating in time

The mean of process is constant



If we have only one realization of the process, there is no means to check ergodicity formally. In practice, we just hope that the process is ergodic